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OPTION PRICING WITH A LEVY-TYPE STOCHASTIC DYNAMIC MODEL FOR STOCK PRICE PROCESS UNDER SEMI-MARKOVIAN STRUCTURAL PERTURBATIONS

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In this work, we consider a stock price process subjected to idiosyncratic Lévy jumps and global structural changes attributed to interventions due to a semi-Markov process. The semi-Markov process decomposes both the time and state domains of the price process into sub-intervals and price state sub-domains respectively, where a Lévy–Itô process operates. The Lévy jumps decompose the space domain of the currently operating Lévy process. We derive an infinitesimal generator for a stock price process and a closed form expression for the conditional characteristic function of a log price. The former result is used to derive a PIDE satisfied by option prices, while the latter could be used to retrieve risk neutral densities via Fourier transform and price European vanilla options. In the sequel, we derive the characteristic function of the residence time of a semi-Markov process. Incompleteness of the market is exhibited through a general change of measure. For pricing purpose, the minimum entropy martingale measure is defined as an Esscher transform.

Keywords: Semi-Markov process; regime switching models; characteristic function; calibration and simulation option prices; minimum entropy.

1. Introduction

The well-known model developed by Black & Scholes (1973), despite its slew of laureates has long shown well documented weaknesses in keeping up with the stylized facts of financial asset returns and option prices. Smiles, smirks and skews are well documented empirical features of implied volatilities (Bulla 2006, Tankov 2003, Elliott *et al.* 2005, Jackson & Jaimungal 2009), unexplained in the context of Black–Scholes model. Moreover, stylized facts of financial time series also cast a doubt on the appropriateness of the normal log return distribution assumption.

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The literature goes about solving these issues in two main ways. The first approach uses time dependent deterministic volatility models (Kruse 2003, Han 2006) to capture most of option market empirical properties. However, it has been shown by González-Uribeaga (2012) that risk neutral volatilities behave in a random manner. This leads to the development of the second modeling approach consisting of stochastic volatility, local volatility and regime switching models. Stochastic volatility models (Hagan *et al.* 2002, Heston 1993) are based on the assumption that volatility is a dynamic process. In local volatility models (Beckers 1980, Dupire 1997), the volatility depends on time and stock price through a deterministic functional. In both cases, in addition to the possibility of a misspecified functional form of the volatility (Chourdakis 2005), the volatility surface often lacks smoothness and at times, takes nonsensical and counterintuitive forms. One of the main advantages of regime switching models as noted by Chourdakis (2005), is the interpretability of the market states while disassociating with the very restrictive functional form assumption of the local and stochastic volatility models.

The present paper is an attempt to extend the latest semi-Markov switching models for stock price and at setting up a general theoretical framework to study qualitative and quantitative properties of asset price processes. To the best of our knowledge, mostly Markovian interventions on price processes are studied in Chan & Zhu (2014b,a), Hainaut & Le Courtois (2014), Hainaut (2011), Momeya (2012), Jackson & Jaimungal (2009) and Chourdakis (2005, 2002). Recently, stochastic models (Preda *et al.* 2014, Swishchuk & Islam 2011, Hunt & Devolder 2011, Hunt & Hahn 2010, Ghosh & Goswami 2009) under the influence of a semi-Markov process have been examined. A Stochastic Maximum Principle for semi-Markov switching jump diffusion models (Deshpande 2014) has been established, leaving out the class of Lévy processes with infinite activity.

The organizational outline of the paper is as follows: In Sec. 2, we introduce the necessary definitions and notations and we present known results. In Sec. 3, we find a closed form solution of a Lévy type of SDE. In Sec. 4, we derive Ito differential formula and the infinitesimal generator for a class of stochastic linear hybrid models under semi-Markovian and Lévy-type structural perturbations. Section 5 is concerned with the derivation of a closed form characteristic function of the log price process. This is useful for recovering risk neutral densities to estimate option prices. Moreover, this provides an alternative tool to the computationally extensive continuous time MCMC and the two-step numerical integration procedure (Hunt & Hahn 2010, Ghosh & Goswami 2009), for simulating option prices and calibrating model parameters to market option prices. In Sec. 6, we exhibit a general change of measure and two risk neutral measures of interest, namely, the minimum entropy martingale measures and the pricing kernel discussed in Siu & Yang (2009). The latter accounts for the regime risk, the jump risk and the Lévy risk. Section 7 is devoted to the presentation of a couple of option price formulas. The first is the well known Fourier transform method based on the method of Carr & Madan (1999). The second formula is a slight modification of the integral formula developed in

Ghosh & Goswami (2009). In Sec. 8, numerical illustrations are given to exhibit the usefulness of the presented results.

2. Preliminary Definitions and Results

In this work, $T^* < \infty$ and $T \in [0, T^*]$. T^* and T stand for the time horizon of the market and the maturity time of a contingent claim, respectively; (Ω, \mathbb{F}, P) is a complete probability space; θ is a semi-Markov process defined on $\mathbb{R}^+ \times (\Omega, \mathbb{F}, P)$ into E , where E is an at most countable subset of the set of natural numbers \mathbb{N} and $\mathbb{R}^+ = [0, \infty)$. For each $n \in \mathbb{N}$, T_n stands for the n th jump time of θ . For $s \in [0, T]$ and $\theta_{s-} = j$, $(L_{s-}^j)_{s \in [0, T]}$ is the Itô Lévy process with small and big jumps G and H and Lévy triplet $(\mu(j), \sigma(j), \nu(j, \cdot))$, where $\mu(j)$, $\sigma(j)$ and $\nu(j, \cdot)$ are the drift rate, the diffusion rate and the Lévy measure, respectively. $(\mathbb{L}_t)_{t \in [0, T]}$, $(\mathbb{H}_t)_{t \in [0, T]}$ and $(\mathbb{B}_t)_{t \in [0, T]}$ are sub-algebras of \mathbb{F} generated by the collection of Lévy processes L^j , $\forall j \in E$, the semi-Markov process θ and $(L_t^\theta, \theta_t)_{t \in [0, T]}$, respectively. Let $(\beta_n)_{n \geq 0}$ and $(\mathbb{B}_t)_{t \in [0, T]}$ be a discrete-time real valued stochastic process and the sub-sigma algebra of \mathbb{F} adapted to the discrete process β_n , respectively. We denote the enlarged filtrations $(\mathbb{L}_t \vee \mathbb{B}_t)_{t \in [0, T]}$ and $(\mathbb{H}_t \vee \mathbb{B}_t)_{t \in [0, T]}$ by $(\bar{\mathbb{L}}_t)_{t \in [0, T]}$ and $(\bar{\mathbb{H}}_t)_{t \in [0, T]}$, respectively. Let $\psi(j, \cdot, \cdot)$ be the Poisson random measure with compensator $\nu(j, dz)$. It is also assumed that the sequence $(\beta_n)_{n \geq 0}$ is independent of both $\psi(j, \cdot, \cdot)$ and the Brownian process B_t , for $j \in E$.

Definition 2.1. (Cinlar 1969) Let θ and $\{T_n\}_{n=1}^\infty$ be a semi-Markov process and its jump time sequence with $T_0 = 0$, respectively. A couple (θ_n, T_n) is called a Markov renewal process with kernel Q induced by the semi-Markov Process (θ_t) , if it satisfies:

$$\begin{aligned} P(\theta_n = j, T_n \leq t \mid (\theta_k, T_k), k = 1, 2, \dots, n-1) \\ = P(\theta_n = j, T_n - T_{n-1} \leq t - T_{n-1} \mid \theta_{n-1}, T_{n-1}) \\ = Q(\theta_{n-1}, j, t - T_{n-1}), \end{aligned} \quad (2.1)$$

where θ_n stands for θ_{T_n} .

Remark 2.1. $\tau_n = T_{n+1} - T_n$ denotes a holding time at T_n . The holding times conditional on the current state are independent Cinlar (1969). The kernel in (2.1) can be represented as

$$Q(i, j, t - T_n) = P(\theta_n = j, T_n - T_{n-1} \leq t - T_{n-1} \mid \theta_{n-1} = i). \quad (2.2)$$

Moreover, for $(\theta_n, T_n) = (\theta_{n(t)}, T_{n(t)})$, $\forall t \in [0, T]$, where

$$n(t) = \max\{k \in \mathbb{N}, T_k \leq t\}. \quad (2.3)$$

In particular,

$$Q(i, j, t) = P(\theta_n = j, T_n - T_{n-1} \leq t \mid \theta_{n-1} = i). \quad (2.4)$$

Furthermore, we define

$$p_{ij} = \lim_{t \rightarrow \infty} Q(i, j, t), \quad (2.5)$$

where p_{ij} is called the steady state transition probability of the embedded Markov chain from state i to state j with $i, j \in E$ and $n(E) = m$.

For the sake of completeness, we present survival distribution and sojourn time distributions associated with the semi-Markov process θ .

Definition 2.2. The conditional cumulative distribution of the holding/sojourn/residence time (respectively, survival function) given that θ transits from a state i to state j is defined by $F(t | i, j) = P(\tau_n \leq t | \theta_n = j, \theta_{n-1} = i)$ (respectively, $S(\cdot | i, j) = 1 - F(\cdot | i, j)$).

In the following lemma, we outline a few well known properties of semi-Markov processes (Cinlar 1969, Ghosh & Goswami 2009).

Lemma 2.1. *The kernel of the semi-Markov process defined in (2.4) is represented by*

$$Q(i, j, t) = p_{i,j} F(t | i, j). \quad (2.6)$$

Moreover,

$$S(t | i) = 1 - \sum_{j \in E} p_{i,j} F(t | i, j), \quad (2.7)$$

$$\frac{f(r | \theta_0 = i)}{P(T_1 > s | \theta_0 = i)} = \frac{-\frac{dS}{dr}(t | i, j)}{S(s^- | i)}, \quad \text{for } r \geq s, \quad (2.8)$$

$$\lambda_{i,j}(t) = p_{i,j} \frac{-\frac{dS}{dt}(t | i, j)}{S(t^- | i)}. \quad (2.9)$$

Proof. We first establish (2.6). From (2.4) we have

$$\begin{aligned} Q(i, j, t) &= P(\theta_n = j, T_n - T_{n-1} \leq t | \theta_{n-1} = i), \\ &= P(\theta_n = j | \theta_{n-1} = i) P(T_n - T_{n-1} \leq t | \theta_{n-1} = i, \theta_n = j) \\ &= p_{i,j} F(t | i, j). \end{aligned}$$

For the proof of (2.7), we use Definition 2.2 and (2.6).

$$\begin{aligned} F(t | i) &= P(\tau_n \leq t | \theta_n = i) \\ &= \sum_{j \in E} p_{i,j} F(t | i, j). \end{aligned} \quad (2.10)$$

Hence,

$$S(t | i) = 1 - \sum_{j \in E} p_{i,j} F(t | i, j).$$

This completes the proof of (2.7). For the proof of (2.8), for $r > s$, we have

$$\begin{aligned}
 & \frac{f(r | \theta_0 = i)}{P(T_1 > s | \theta_0 = i)}, \quad (\text{definition of conditional density}) \\
 &= \frac{-\frac{dS}{dr}(r | i)}{P(T_1 > s | \theta_s = i)}, \quad (\text{definition of survival function}) \\
 &= \frac{-\frac{dS}{dr}(r | i)}{S(s^- | i)}. \tag{2.11}
 \end{aligned}$$

This completes the proof of (2.8). The proof of (2.9) follows from the definition of Hazard functions and (2.6). This completes the proof of the lemma. \square

Remark 2.2. A homogeneous Markov process is a particular case of semi-Markov process. Hence, $Q_{ij}(t) = p_{ij}(1 - e^{q(i)})$. We also have the following relationship:

$$q_{ij} = p_{ij}q(i), \tag{2.12}$$

where $(q_{ij})_{m \times m}$ is the Infinitesimal generator (intensity matrix) of a Markov process, and $(p_{ij})_{m \times m}$ is its transition probability matrix defined in (2.5). The following can also be inferred from (2.6) and (2.7):

$$S(t | i) = 1 - \sum_{j \in E} Q(i, j, t). \tag{2.13}$$

Definition 2.3. Let y_t be the backward recurrence time of the semi-Markov process θ at time t . y_t is defined as follows:

$$y_t = \sum_{n \geq 0} (t - T_n) 1_{(T_n \leq t < T_{n+1})}, \tag{2.14}$$

where the sequence $\{T_n\}_{n=0}^\infty$ is introduced in Definition 2.1.

Definition 2.4. Let $\psi : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{R}^+ \mapsto \mathbb{R}$ be the random Poisson measure with intensity measure ν , H and G smooth functions defined on $\mathbb{R}^+ \times \mathbb{R}$ into \mathbb{R} , with G satisfying the condition: $\int_{z \in \mathbb{R}} ((1 + H^2(z, j)) 1_{|z| > 1} + G^2(z, j) 1_{|z| \leq 1}) \nu(j, dz) < \infty$, $\forall j \in E$. Moreover, $\bar{\psi} = \psi - \nu$ denotes the compensated Poisson measure associated with ψ .

In the following we present a lemma, which would be used, subsequently.

Lemma 2.2. Let (a_n, b_n) and (c_n, d_n) be two renewal processes defined on the same probability space (Ω, \mathbb{F}, P) and state space E . Then the renewal processes have identical transition probability matrices and sojourn time distributions, respectively.

Proof. From (2.5), it is clear that the transition probability and the holding time distribution are completely defined by the kernel matrix. In fact, we have,

$$\lim_{t \rightarrow \infty} Q(i, j, t) = \lim_{t \rightarrow \infty} p_{ij} F(t | i, j) = p_{ij},$$

and hence,

$$F(t | i, j) = \frac{Q(i, j, t)}{p_{ij}}.$$

This establishes the result. \square

Lemma 2.3. *Let $n(t)$ be defined as in (2.3). The pair $(\theta_t, t - T_{n(t)})$ is a Markov process.*

Proof. Let be $s \leq t$ with $t \in [T_n, T_{n+1})$ and $s \in [T_m, T_{m+1})$ ($m < n$). For $u \leq s$, we have:

$$\begin{aligned} P(\theta_t = i, t - T_{n(t)} \leq a | (\theta_u, u - T_{n(u)})), \quad (\text{for some } a \in \mathbb{R}^+) \\ = P(\theta_{T_n} = i, T_{n+1} - T_n \leq a | (\theta_u, u - T_{n(u)})), \quad (\text{for } t \in [T_n, T_{n+1})) \\ = P(\theta_{T_n} = i, T_{n+1} - T_n \leq a | (\theta_{T_k}, T_k), k \leq m), \quad (\text{for } s \in [T_m, T_{m+1})) \\ = P(\theta_{T_n} = i, T_{n+1} - T_n \leq a | \theta_{T_m}), \quad (\text{Markov renewal process property}) \\ = P(\theta_t = i, t - T_{n(t)} \leq a | \theta_s), \quad (\text{definition of } n(t)). \end{aligned}$$

Hence, the probability at a future time depends only on the most current information at time s . This shows that $(\theta_t, t - T_{n(t)})$ is a Markov process. \square

Remark 2.3. For the remainder of this paper θ is a semi-Markov process with jump time T_n , with sojourn time $\tau_n = T_{n+1} - T_n \sim f(|\theta_n, \theta_{n+1})$ with CDF $F(|\theta_n, \theta_{n+1})$ and with survival CDF $S(|\theta_n, \theta_{n+1}) = 1 - F(|\theta_n, \theta_{n+1})$. The semi-Markov kernel is denoted $Q(i, j, t)$, the backward recurrence time y_t is defined in (2.14) and $(p_{i,j})_{m \times m}$ is the transition probability matrix of the embedded Markov chain.

3. Method for Finding Closed Form Solutions

In this section, we find a closed form solution of a Lévy-type Linear Stochastic Differential Equation under semi-Markovian structural perturbations. The presented extension is based on the procedure described in Ladde & Ladde (2013). The usefulness of the result is at least two-fold. It is used to establish the martingale property for certain processes in Sec. 6. In addition, it is used to formulate a general expression for the simple return process with any Lévy and semi-Markov jump choices. We consider the following Lévy-type SDE:

$$dx_t = x_t - dL_t^\theta, \quad x(0) = x_0, \quad (3.1)$$

where

$$\begin{aligned} dL_t^\theta = \mu(\theta_t)dt + \sigma(\theta_t)dB_t + \int_{|z|>1} H(z, \theta_t)\psi(\theta_t, dz, dt) \\ + \int_{|z|\leq 1} G(z, \theta_t)\bar{\psi}(\theta_t, dz, dt), \end{aligned} \quad (3.2)$$

θ is the semi-Markov process defined in Sec. 2; ψ , ν G and H are in Definition 2.4. Following the procedure described in Chapter 2, Ladde & Ladde (2013), we break down (3.1) into the following four types of simplified SDEs:

$$\begin{cases} dx_t^1 = x_t^1 \mu(\theta_t) dt \\ dx_t^2 = x_t^2 \sigma(\theta_t) dB_t \\ dx_t^3 = x_t^3 \int_{|z|>1} H(z, \theta_t) \psi(\theta_t, dz, dt) \\ dx_t^4 = x_t^4 \int_{|z|\leq 1} G(z, \theta_t) \bar{\psi}(\theta_t, dz, dt). \end{cases} \quad (3.3)$$

Imitating the procedure in Ladde & Ladde (2013), the closed form solution processes of

$$dx_t^1 = x_t^1 \mu(\theta_t) dt \quad \text{and} \quad dx_t^2 = \sigma(\theta_t) x_t^2 dB_t$$

are

$$\begin{aligned} x_t^1 &= \left[\exp \left(\int_0^t \mu(\theta_s) ds \right) \right] c_1 \quad \text{and} \\ x_t^2 &= \exp \left[-\frac{1}{2} \int_0^t \sigma^2(\theta_s) ds + \int_0^t \sigma(\theta_s) dB_s \right] c_2, \end{aligned} \quad (3.4)$$

respectively; c_1 and c_2 are arbitrary constants. We next consider the third type of SDE in (3.3)

$$dx_t^3 = x_t^3 \int_{|z|>1} H(z, \theta_t) \psi(\theta_t, dz, dt). \quad (3.5)$$

We seek a solution of a form

$$x_t^3 = \exp \left[\int_0^t \int_{|z|>1} f_4(z, \theta_s) \psi(\theta_s, dz, ds) \right] c_3, \quad (3.6)$$

where f_4 is an unknown smooth function to be determined, and c_3 is a real random variable. The Ito integral for pure jump processes (3.6) yields

$$\begin{aligned} x_{t+\Delta t}^3 - x_t^3 &= \sum_{t \leq s \leq t+\Delta t} (x_s^3 - x_{s-}^3) \\ &= \int_t^{t+\Delta t} \int_{|z|>1} (x_{s-}^3 e^{f_4(z, \theta_{s-})} - x_{s-}^3) \psi(\theta_{s-}, dz, ds) \\ &= \int_t^{t+\Delta t} x_{s-}^3 \int_{|z|>1} (e^{f_4(z, \theta_{s-})} - 1) \psi(\theta_{s-}, dz, ds). \end{aligned} \quad (3.7)$$

As Δt becomes very small, (3.7) reduces to

$$dx_t^3 = x_t^3 \int_{|z|>1} (e^{f_4(z, \theta_t)} - 1) \psi(\theta_t, dz, dt). \quad (3.8)$$

Since x^3 is solution of stochastic differential equation (3.5), we repeat the procedure described in Ladde & Ladde (2013) and obtain

$$\exp(f_4(z, \theta_t)) - 1 = H(z, \theta_t). \quad (3.9)$$

Hence,

$$f_4(z, \theta_t) = \ln(1 + H(z, \theta_t)). \quad (3.10)$$

Therefore, the general solution of (3.5) is represented by

$$x_t^3 = \exp \left[\int_0^t \int_{|z|>1} \ln(1 + H(z, \theta_s)) \psi(\theta_s, dz, ds) \right] c_3. \quad (3.11)$$

x^3 is almost surely finite. Finally, we find a solution of the following stochastic differential equation

$$dx_t^4 = x_t^4 - \int_{|z|\leq 1} G(z, \theta_t) \bar{\psi}(\theta_t, dz, dt). \quad (3.12)$$

We seek a solution process of (3.12) in the following form:

$$x_t^4 = \exp \left[\int_0^t \int_{|z|\leq 1} f_5(z, \theta_s) \bar{\psi}(\theta_s, dx, ds) + \int_0^t \int_{|z|\leq 1} f_6(z, \theta_s) \nu(\theta_s, dz) ds \right] c_4, \quad (3.13)$$

where f_5 and f_6 are unknown smooth functions to be determined, and c_4 is a real valued random variable. x^4 in (3.12) is an exponential function semi-martingale of the form $v = \int_0^t \int_{|z|\leq 1} f_5(z, \theta_s) \bar{\psi}(\theta_s, dx, ds) + \int_0^t \int_{|z|\leq 1} f_6(z, \theta_s) \nu(\theta_s, dz) ds$. Applying the Ito formula for discontinuous semi-martingales, Applebaum (2009), we have

$$\begin{aligned} dx_t^4 &= \frac{\partial x_t^4}{\partial v} dv^c + \frac{1}{2} \frac{\partial^2 x_t^4}{\partial v^2} d(v^c) d(v^c) + \left(\Delta x_t^4 - \frac{\partial x_t^4}{\partial L} \Delta L \right) \\ &= x_{t-}^4 \int_{|z|\leq 1} (f_6(z, \theta_{t-}) + \exp(f_5(z, \theta_{t-})) - 1 - f_5(z, \theta_{t-})) \nu(\theta_{t-}, dz) dt \\ &\quad + x_{t-}^4 \int_{|z|\leq 1} (\exp(f_5(z, \theta_{t-})) - 1) \bar{\psi}(\theta_{t-}, dz, dt). \end{aligned} \quad (3.14)$$

Again, following the procedure for finding solution processes in Ladde and Ladde (2013), we get

$$\begin{cases} f_6(z, \theta_t) + \exp(f_5(z, \theta_t)) - 1 - f_5(z, \theta_t) = 0 \\ \exp(f_5(z, \theta_t)) - 1 = G(z, \theta_t). \end{cases} \quad (3.15)$$

Hence,

$$\begin{cases} f_5(z, \theta_t) = \ln(1 + G(z, \theta_t)) \\ f_6(z, \theta_t) = \ln(1 + G(z, \theta_t)) - G(z, \theta_t). \end{cases} \quad (3.16)$$

Therefore,

$$\begin{aligned}
 x_t^4 = \exp & \left[\int_0^t \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) ds \right. \\
 & \left. + \int_0^t \int_{|z| \leq 1} \ln(1 + G(z, \theta_{s-})) \bar{\psi}(\theta_{s-}, dz, ds) \right] c_4.
 \end{aligned} \tag{3.17}$$

The product of x^1 and x^2 from (3.4) analog with x^3 and x_4 in (3.11) and (3.17), respectively, yields the solution of initial value problem (3.1)

$$\begin{aligned}
 x_t = x_0 \exp & \left[\int_0^t \left[\mu(\theta_{s-}) - \frac{1}{2} \sigma^2(\theta_{s-}) \right. \right. \\
 & \left. + \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) \right] ds \\
 & + \int_0^t \sigma(\theta_{s-}) dB_s + \int_0^t \int_{|z| \leq 1} \ln(1 + G(z, \theta_{s-})) \bar{\psi}(\theta_{s-}, dz, ds) \\
 & \left. + \int_0^t \int_{|z| > 1} \ln(1 + H(z, \theta_{s-})) \psi(\theta_{s-}, dz, ds) \right].
 \end{aligned} \tag{3.18}$$

In the following, we present a few versions of (3.18).

Remark 3.1. We note that adding and subtracting

$$\int_0^t \int_{|z| \leq 1} G(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) \quad \text{and} \quad \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \psi(\theta_{s-}, dz, ds),$$

(3.18) reduces to

$$\begin{aligned}
 x_t = x_0 \exp & \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s + \int_0^t \int_{|z| \leq 1} G(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) \right. \\
 & + \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \psi(\theta_{s-}, dz, ds) - \frac{1}{2} \int_0^t \sigma^2(\theta_{s-}) ds \\
 & + \int_0^t \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) ds \\
 & + \int_0^t \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \bar{\psi}(\theta_{s-}, dz) \\
 & \left. + \int_0^t \int_{|z| > 1} [\ln(1 + H(z, \theta_{s-})) - H(z, \theta_{s-})] \psi(\theta_{s-}, dz, ds) \right].
 \end{aligned} \tag{3.19}$$

Moreover, adding and subtracting

$$\begin{aligned} & \int_0^t \int_{|z| \leq 1} G(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds), \\ & \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds), \quad \text{and} \\ & \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \nu(\theta_{s-}, dz) ds, \end{aligned}$$

(3.18) becomes

$$\begin{aligned} x_t = x_0 \exp & \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s + \int_0^t \int_{|z| \leq 1} G(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) \right. \\ & + \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) \\ & + \int_0^t \int_{|z| > 1} \ln(H(z, \theta_{s-}) + 1) \nu(\theta_{s-}, dz) ds - \frac{1}{2} \int_0^t \sigma^2(\theta_{s-}) ds \\ & + \int_0^t \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \bar{\psi}(\theta_{s-}, dz, ds) \\ & + \int_0^t \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) ds \\ & \left. + \int_0^t \int_{|z| > 1} [\ln(1 + H(z, \theta_{s-})) - H(z, \theta_{s-})] \bar{\psi}(\theta_{s-}, dz, ds) \right]. \end{aligned} \quad (3.20)$$

In the following remark, we take a look at a few particular cases of interest which will be used, subsequently.

Remark 3.2. If $H(z, \theta_s)$, $G(z, \theta_s)$ and L_s^θ in (3.2) are replaced by $e^{H(z, \theta_s)} - 1$, $e^{G(z, \theta_s)} - 1$ and

$$\begin{aligned} dL_s^\theta = & \mu(\theta_s) ds + \sigma(\theta_s) dB_s + \int_{|z| \leq 1} [e^{G(z, \theta_s)} - 1] \bar{\psi}(\theta_s, dz, ds) \\ & + \int_{|z| > 1} [e^{H(z, \theta_s)} - 1] \psi(\theta_s, dz, ds), \end{aligned} \quad (3.21)$$

respectively, then the solution of the IVP (3.1) in (3.18), (3.19) and (3.20) reduce to

$$\begin{aligned} x_t = x_0 \exp & \left[\int_0^t \left[\mu(\theta_{s-}) - \frac{1}{2} \sigma^2(\theta_{s-}) \right. \right. \\ & \left. \left. + \int_{|z| \leq 1} [G(z, \theta_{s-}) + 1 - e^{G(z, \theta_{s-})}] \nu(\theta_{s-}, dz) \right] ds \right] \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t \sigma(\theta_{s-}) dB_s + \int_0^t \int_{|z| \leq 1} G(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \psi(\theta_{s-}, dz, ds) \Big], \tag{3.22}
 \end{aligned}$$

$$\begin{aligned}
 x_t = x_0 \exp & \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s \right. \\
 & + \int_0^t \int_{|z| \leq 1} [e^{G(z, \theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_0^t \int_{|z| > 1} [e^{H(z, \theta_{s-})} - 1] \psi(\theta_{s-}, dz, ds) - \frac{1}{2} \int_0^t \sigma^2(\theta_{s-}) ds \\
 & + \int_0^t \int_{|z| \leq 1} [G(z, \theta_{s-}) - e^{G(z, \theta_{s-})} + 1] \nu(\theta_{s-}, dz) ds \\
 & + \int_0^t \int_{|z| \leq 1} [G(z, \theta_{s-}) - e^{G(z, \theta_{s-})} + 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & \left. + \int_0^t \int_{|z| > 1} [H(z, \theta_{s-}) - e^{H(z, \theta_{s-})} + 1] \psi(\theta_{s-}, dz, ds) \right], \tag{3.23}
 \end{aligned}$$

$$\begin{aligned}
 x_t = x_0 \exp & \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s \right. \\
 & + \int_0^t \int_{|z| \leq 1} [e^{G(z, \theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_0^t \int_{|z| > 1} [e^{H(z, \theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \nu(\theta_{s-}, dz) ds - \frac{1}{2} \int_0^t \sigma^2(\theta_{s-}) ds \\
 & + \int_0^t \int_{|z| \leq 1} [G(z, \theta_{s-}) - e^{G(z, \theta_{s-})} + 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_0^t \int_{|z| \leq 1} [G(z, \theta_{s-}) - e^{G(z, \theta_{s-})} + 1] \nu(\theta_{s-}, dz) ds \\
 & \left. + \int_0^t \int_{|z| > 1} [H(z, \theta_{s-}) - e^{H(z, \theta_{s-})} + 1] \bar{\psi}(\theta_{s-}, dz, ds) \right]. \tag{3.24}
 \end{aligned}$$

In addition, if $\mu(\theta_s)$ in (3.2) is replaced by $[\mu(\theta_s) + \frac{1}{2}\sigma^2(\theta_s) + \int_{|z|\leq 1}[e^{G(z,\theta_s)} - 1 - G(z,\theta_s)]\nu(\theta_s, dz)]$, then (3.22), (3.23) and (3.24), respectively, reduce to

$$\begin{aligned} x_t &= x_0 \exp \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s \right. \\ &\quad + \int_0^t \int_{|z|\leq 1} G(\theta_{s-}, z) \bar{\psi}(\theta_{s-}, dz, ds) \\ &\quad \left. + \int_0^t \int_{|z|>1} H(\theta_{s-}, z) \psi(\theta_{s-}, dz, ds) \right] \\ &= x_0 \exp[L_t^\theta], \end{aligned} \tag{3.25}$$

$$\begin{aligned} x_t &= x_0 \exp \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s \right. \\ &\quad + \int_0^t \int_{|z|\leq 1} [e^{G(z,\theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\ &\quad + \int_0^t \int_{|z|>1} [e^{H(z,\theta_{s-})} - 1] \psi(\theta_{s-}, dz, ds) \\ &\quad + \int_0^t \int_{|z|\leq 1} [G(z, \theta_{s-}) - e^{G(z,\theta_{s-})} + 1] \bar{\psi}(\theta_{s-}, dz) \\ &\quad \left. + \int_0^t \int_{|z|>1} [H(z, \theta_{s-}) - e^{H(z,\theta_{s-})} + 1] \psi(\theta_{s-}, dz, ds) \right], \end{aligned}$$

$$\begin{aligned} x_t &= x_0 \exp \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s \right. \\ &\quad + \int_0^t \int_{|z|\leq 1} [e^{G(z,\theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\ &\quad + \int_0^t \int_{|z|>1} [e^{H(z,\theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\ &\quad + \int_0^t \int_{|z|>1} [e^{H(z,\theta_{s-})} - 1] \nu(\theta_{s-}, dz) ds \\ &\quad + \int_0^t \int_{|z|\leq 1} [G(z, \theta_{s-}) - e^{G(z,\theta_{s-})} + 1] \psi(\theta_{s-}, dz, ds) \\ &\quad \left. + \int_0^t \int_{|z|>1} [H(z, \theta_{s-}) - e^{H(z,\theta_{s-})} + 1] \bar{\psi}(\theta_{s-}, dz, ds) \right], \end{aligned} \tag{3.26}$$

where L^θ is defined in (3.2). Moreover, if $\mu(\theta_{t-})$ in (3.1) is replaced by $[\mu(\theta_{s-}) + \frac{1}{2}\sigma^2(\theta_{s-}) + \int_{|z|\leq 1}[e^{G(z,\theta_{s-})} - 1 - G(z, \theta_{s-})]\nu(\theta_{s-}, dz) + \int_{|z|>1} H(z, \theta_{s-})\nu(\theta_{s-}, dz)]$,

then (3.24) reduces to

$$\begin{aligned}
 x_t = x_0 \exp & \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s \right. \\
 & + \int_0^t \int_{|z| \leq 1} [e^{G(z, \theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_0^t \int_{|z| > 1} [e^{H(z, \theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_0^t \int_{|z| \leq 1} [G(z, \theta_{s-}) - e^{G(z, \theta_{s-})} + 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & \left. + \int_0^t \int_{|z| > 1} [H(z, \theta_{s-}) - e^{H(z, \theta_{s-})} + 1] \bar{\psi}(\theta_{s-}, dz, ds) \right]. \quad (3.27)
 \end{aligned}$$

4. Ito Differential Formula

In this section, we define the asset price model and we derive the infinitesimal generator of the quadruplet (t, y, θ, x) . We denote L^θ the Itô Lévy process with small and big jumps G and H and Lévy triplet $(\mu(\theta_t), \sigma(\theta_t), \nu(\theta_t, dz))$ defined (3.2). Following the argument used in Ladde & Ladde (2013), we define a linear stochastic hybrid dynamic model for stock price process under structural perturbations of semi-Markov and Lévy processes.

Definition 4.1. A linear stochastic hybrid dynamic model under semi-Markov and Lévy structural perturbations is defined as follows:

$$\begin{cases} dx(t) = x(t^-) dL_t^{\theta_n}, & x(T_n) = x_n, & t \in [T_n, T_{n+1}) \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}), & x(0) = x_0, & n \in I(1, \infty) = \mathbb{N}, \end{cases} \quad (4.1)$$

where $\{T_n\}_{n=1}^\infty$ is an increasing sequence of jump/regime switching times of the semi-Markov process θ with $T_0 = 0$ introduced in Definition 2.1; for $n \in I(0, \infty) = \{0, 1, 2, 3, \dots\}$, β_n denotes the discrete time state jump process caused by the semi-Markov process from state θ_{n-1} at T_{n-1} to θ_n at T_n ; it is denoted $\beta_n = \beta_{\theta_{n-1}, \theta_n}$, highlighting the assumption that semi-Markov jump distributions depend only on the previous and current market states. The density function of $\beta_{i,j}$ is $b(\cdot | i, j)$ and L_t^θ is defined in (3.2).

Remark 4.1. A few observations about the model in the context of Remark 3.2 are in order. The solution of (4.1) can be described by the following discrete time iterative process Ladde & Ladde (2013)

$$\begin{cases} x(t, T_n, x_n) = x_n \exp \left[\int_{T_n}^t dL_s^{\theta_n} \right], & t \in [T_n, T_{n+1}) \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}), \end{cases} \quad (4.2)$$

where L_t^θ is defined as the exponent of the solution process of (3.1) as expressed in (3.25). The semi-Markov process decomposes both the time and state domains causing structural changes in the stock price process, while the Lévy process directly decomposes the state domain of definition of the stochastic dynamic model.

Remark 4.2. From (3.25) and (4.1), the size of the jump in log price at time T_n is $\ln(\beta_n)$. The density function of $\ln(\beta_n)$ is described by

$$\bar{b}(z | \theta_{n-1}, \theta_n) = b(e^z | \theta_{n-1}, \theta_n) e^z, \quad (4.3)$$

where $b(\cdot | \theta_{n-1}, \theta_n)$ denotes the density of β_n and e is the Naperian base. We further note that the discrete time dynamic system in (4.2) is an intervention process. A feature of interest of this model is its potential to capture, simultaneously, three important stylized facts. The volatility clustering exhibited in log return time series, the slowly decaying autocorrelation of square returns and the observed correlation between log returns and volatility (Chourdakis 2005, Bulla 2006). As the market switches from one state to another, the diffusion rate changes while the asset price is subjected to a jump. Thus the diffusion rate and the price jumps are modulated by the process θ .

For the development of an infinitesimal generator, in the following, we define a point process encoding both the regime switches and the jumps of x at regime switches. At each regime change, we note that the jump in log price is $\ln(\beta_n)$. We define $E^2 = \{(i, j), (i, j) \in E \times E, i \neq j\}$ and the power set of E^2 , $\mathcal{P}(E^2)$. $\mathbb{B}(\mathbb{R})$ is the Borel sigma algebra of the real line. We are ready to define the aforementioned point process.

Definition 4.2. β_n and θ_n are introduced in Definition 4.1. Let $N(t, A, B)$ be a stochastic process defined on $[0, T] \times \mathbb{B}(\mathbb{R}) \times \mathcal{P}(E^2)$ into $(0, \infty)$ as

$$N(t, A, B) = \sum_{n \geq 1} 1_{(t \geq T_n, \ln(\beta_n) \in A, (\theta_{n-1}, \theta_n) \in B)}, \quad (4.4)$$

and $N(t, A, B)$ stands for the number of regime switches in B with corresponding log price jumps $\ln(\beta_n) \in A$ by time t .

Remark 4.3. We observe that

$$N(t, A, B) = \sum_{(i,j) \in B} N(t, A, \{(i, j)\}), \quad (4.5)$$

where $N(t, A, \{(i, j)\})$ counts the number of regime switches from i to j with corresponding log price jump $\ln(\beta_n) \in A$.

In the following lemma, we derive the predictable compensator process for $N(t, A, \{(i, j)\})$.

Lemma 4.1. *Let $N(t, A, \{(i, j)\})$ be the point process introduced in Definition 4.2. Then*

$$N(t \wedge T_n, A, \{(i, j)\}) - \gamma(t \wedge T_n, A, \{(i, j)\}) \quad (4.6)$$

is a martingale with respect to the filtration $(\bar{\mathbb{H}}_t)_{t \geq 0}, \forall n \in I(1, \infty)$, where

$$\gamma(t, A, \{(i, j)\}) = \int_0^t \int_{z \in A} \bar{b}(z | i, j) \lambda_{i,j}(y_s) dz ds, \quad (4.7)$$

and $\lambda_{i,j}$ are defined in (2.9).

Proof. From Brémaud (1981), Siu & Ladde (2011), it is enough to prove that $N(t \wedge T_n, A, \{(i, j)\}) - \gamma(t \wedge T_n, A, \{(i, j)\})$ is an $(\bar{\mathbb{H}}_s)_{s \geq 0}$ -martingale. For any $0 \leq s \leq t$ and for each $n \in I(1, \infty)$, it satisfies

$$\begin{aligned} E[N(t \wedge T_n, A, \{(i, j)\}) - \gamma(t \wedge T_n, A, \{(i, j)\})] \\ - [N(s \wedge T_n, A, \{(i, j)\}) - \gamma(s \wedge T_n, A, \{(i, j)\})] | \bar{\mathbb{H}}_s = 0 \end{aligned} \quad (4.8)$$

and if and only if

$$\begin{aligned} E(N(t \wedge T_n, A, \{(i, j)\}) - N(s \wedge T_n, A, \{(i, j)\}) | \bar{\mathbb{H}}_s) \\ = E(\gamma(t \wedge T_n, A, \{(i, j)\}) - \gamma(s \wedge T_n, A, \{(i, j)\}) | \bar{\mathbb{H}}_s). \end{aligned} \quad (4.9)$$

We prove that (4.9) holds. We first prove that (4.9) holds when the jump process N is stopped at T_1 . We then prove by the Principle of Mathematical Induction that (4.9) is true when N is stopped at time T_n . From Definition 4.2, (2.5) and for $0 \leq s \leq t$, we have

$$\begin{aligned} E(N(t \wedge T_1, A, \{(i, j)\}) - N(s \wedge T_1, A, \{(i, j)\}) | \bar{\mathbb{H}}_s) \\ = \begin{cases} E(1_{(T_1 \leq t, \ln(\beta_1) \in A, \theta_1 = j, \theta_0 = i)} - 1_{(T_1 \leq s, \ln(\beta_1) \in A, \theta_1 = j, \theta_0 = i)} | \theta_0, T_1 > s), & \text{for } T_1 > s \\ 0, & \text{for } T_1 \leq s, \end{cases} \\ = 1_{(T_1 > s)} E(1_{(T_1 \leq t, \ln(\beta_1) \in A, \theta_1 = j, \theta_0 = i)} - 1_{(T_1 \leq s, \ln(\beta_1) \in A, \theta_1 = j, \theta_0 = i)} | \theta_0, T_1 > s) \\ = 1_{(T_1 > s)} E(1_{(s \leq T_1 \leq t, \ln(\beta_1) \in A, \theta_1 = j, \theta_0 = i)} | \theta_0, T_1 > s) \\ = 1_{(T_1 > s)} 1_{(\theta_0 = i)} \frac{P(s \leq T_1 \leq t, \ln(\beta_1) \in A, \theta_1 = j | \theta_0 = i)}{P(T_1 > s | \theta_0 = i)} \\ = 1_{(T_1 > s)} 1_{(\theta_0 = i)} p_{ij} \frac{P(s \leq T_1, \ln(\beta_1) \in A | \theta_1 = j, \theta_0 = i)}{S(s | i)} \\ - 1_{(T_1 > s)} 1_{(\theta_0 = i)} p_{ij} \frac{P(t \leq T_1, \ln(\beta_1) \in A | \theta_1 = j, \theta_0 = i)}{S(s | i)} \end{aligned}$$

$$\begin{aligned}
&= 1_{(T_1 > s)} 1_{(\theta_0 = i)} P(\ln(\beta_1) \in A \mid \ln(\beta_1)i, j) p_{ij} \frac{P(s \leq T_1 \mid \theta_0 = i, \theta_1 = j)}{S(s \mid i)} \\
&\quad - 1_{(T_1 > s)} 1_{(\theta_0 = i)} P(\ln(\beta_1) \in A \mid \ln(\beta_1)i, j) p_{ij} \frac{P(t \leq T_1 \mid \theta_0 = i, \theta_1 = j)}{S(s \mid i)} \\
&= 1_{(T_1 > s)} 1_{(\theta_0 = i)} p_{ij} P(\ln(\beta_1) \in A \mid \theta_0 = i, \theta_1 = j) \frac{-\Delta S(t \mid i, j)}{S(s \mid i)}, \tag{4.10}
\end{aligned}$$

where $\Delta S(t \mid i, j) = S(t \mid \theta_0 = i, \theta_1 = j) - S(s \mid \theta_0 = i, \theta_1 = j)$, with $S(\cdot \mid i, j)$ denoting the conditional survival distribution of sojourn time when the process switches from i to j . From (2.8), (2.9) and (4.3), we have

$$\begin{aligned}
&E[N(t \wedge T_1, A, \{i, j\}) - N(s \wedge T_1, A, \{i, j\})] \\
&= \int_s^t \int_{z \in A} 1_{(T_1 > s)} 1_{(\theta_0 = i)} \bar{b}(z \mid i, j) \lambda_{i,j}(y_u) du dz. \tag{4.11}
\end{aligned}$$

On the other hand, from (4.7) and (2.8), we obtain

$$\begin{aligned}
&E[\gamma(t \wedge T_1, A, \{(i, j)\}) - \gamma(s \wedge T_1, A, \{(i, j)\}) \mid \bar{\mathbb{H}}_s] \\
&= 1_{(\theta_0 = i)} 1_{T_1 > s} E \left[\int_{z \in A} \int_0^{t \wedge T_1} \bar{b}(z \mid i, j) \lambda_{i,j}(y_u) du dz \right. \\
&\quad \left. - \int_{z \in A} \int_0^{s \wedge T_1} \bar{b}(z \mid i, j) \lambda_{i,j}(y_u) du dz \mid \bar{\mathbb{H}}_s \right] \\
&= 1_{T_1 > s} 1_{(\theta_0 = i)} E \left[\int_{T_1 \wedge s}^{T_1 \wedge t} P(\ln(\beta_1) \in A \mid i, j) \lambda_{i,j}(y_u) du \mid T_1 > s, \theta_0 = i \right], \\
&\hspace{25em} \text{(Fubini's theorem)} \\
&= 1_{T_1 > s} 1_{(\theta_0 = i)} \int_s^\infty \int_{T_1 \wedge s}^{T_1 \wedge t} P(\ln(\beta_1) \in A \mid i, j) \lambda_{i,j}(y_u) du \frac{-dS(r \mid \theta_0 = i)}{S(s \mid \theta_0 = i)} \\
&= 1_{T_1 > s} 1_{(\theta_0 = i)} \left[\int_s^t \int_{r \wedge s}^{r \wedge t} P(\ln(\beta_1) \in A \mid i, j) \lambda_{i,j}(y_u) du \frac{-dS(r \mid \theta_0 = i)}{S(s \mid \theta_0 = i)} \right. \\
&\quad \left. + \int_t^\infty \int_{r \wedge s}^{r \wedge t} P(\ln(\beta_1) \in A \mid i, j) \lambda_{i,j}(y_u) du \frac{-dS(r \mid \theta_0 = i)}{S(s \mid \theta_0 = i)} \right] \\
&= 1_{T_1 > s} 1_{(\theta_0 = i)} \left[-\frac{1}{S(s \mid i, j)} \int_s^t \int_s^r P(\ln(\beta_1) \in A \mid i, j) \lambda_{i,j}(y_u) du dS(r \mid \theta_0) \right. \\
&\quad \left. + \int_t^\infty \left[\int_s^t P(\ln(\beta_1) \in A \mid i, j) \lambda_{i,j}(y_u) du \right] \frac{-dS(r \mid \theta_0 = i)}{S(s \mid \theta_0 = i)} \right]
\end{aligned}$$

$$\begin{aligned}
 &= 1_{T_1 > s} 1_{(\theta_0 = i)} \left[-\frac{1}{S(s | \theta_0 = i)} \int_s^t \int_s^r P(\ln(\beta_1) \in A | i, j) \lambda_{i,j}(y_u) dS(r | \theta_0) du \right. \\
 &\quad \left. + \int_s^t \left[\underbrace{\int_t^\infty \frac{-dS(r | \theta_0 = i)}{S(s | \theta_0 = i)}}_{\text{}} \right] P(\ln(\beta_1) \in A | i, j) \lambda_{i,j}(y_u) du \right] \\
 &= 1_{T_1 > s} 1_{(\theta_0 = i)} \left[-\frac{1}{S(s | \theta_0 = i)} \int_s^t P(\ln(\beta_1) \in A | i, j) \lambda_{i,j}(y_u) \right. \\
 &\quad \times \left[\int_u^t dS(r | \theta_0 = i) \right] du \\
 &\quad \left. + \frac{S(t | \theta_0 = i)}{S(s | \theta_0 = i)} \int_{[s,t]} P(\ln(\beta_1) \in A | i, j) \lambda_{i,j}(y_u) du \right] \\
 &= 1_{T_1 > s} 1_{(\theta_0 = i)} P(\ln(\beta_1) \in A | i, j) \left[\frac{1}{S(s | \theta_0 = i)} \int_s^t \lambda_{i,j}(y_u) [S(u | \theta_0 = i) \right. \\
 &\quad \left. - S(t | \theta_0 = i)] du + \frac{S(t | \theta_0 = i)}{S(s | \theta_0 = i)} \int_s^t \lambda_{i,j}(y_u) du \right] \\
 &= 1_{T_1 > s} 1_{(\theta_0 = i)} \frac{P(\ln(\beta_1) \in A | i, j)}{S(s | \theta_0 = i)} \int_s^t \lambda_{i,j}(u) S(u | \theta_0 = i) du \\
 &= 1_{T_1 > s} 1_{(\theta_0 = i)} p_{ij} \frac{P(\ln(\beta_1) \in A | i, j)}{S(s | \theta_0 = i)} \int_s^t S(u | \theta_0 = i) \frac{-dS(u | i, j)}{S(u | \theta_0 = i)} \\
 &= 1_{T_1 > s} 1_{(\theta_0 = i)} p_{ij} P(\ln(\beta_1) \in A | i, j) \frac{S(s | i, j) - S(t | i, j)}{S(s | i)}. \tag{4.12}
 \end{aligned}$$

From (4.10) and (4.12), we get

$$\begin{aligned}
 &E(\gamma(t \wedge T_1, A, \{(i, j)\}) - \gamma(s \wedge T_1, A, \{(i, j)\}) | \bar{\mathbb{H}}_s) \\
 &= 1_{T_1 > s} 1_{(\theta_0 = i)} p_{ij} P(\ln(\beta_1) \in A | i, j) \frac{S(s | i, j) - S(t | i, j)}{S(s | \theta_0)} \\
 &= E(N(t \wedge T_1, A, \{(i, j)\}) - N(s \wedge T_1, A, \{(i, j)\}) | \bar{\mathbb{H}}_s).
 \end{aligned}$$

This establishes (4.9). Hence, the stopped point process $N(t \wedge T_1, A, \{(i, j)\})$ has predictable compensator $\gamma(t \wedge T_1, A \times \{(i, j)\})$ defined in (4.7). Assuming that (4.8) is valid for some $k \in I(1, \infty)$, and repeating the above argument, we verify the induction assumption. By the principle of mathematical induction, we conclude that $N((t \wedge T_k, t \wedge T_{k+1}], A, \{(i, j)\}) - \gamma(t \wedge T_k, t \wedge T_{k+1}], A, \{(i, j)\})$ is an $(\bar{\mathbb{H}}_t)_{t > 0}$ -martingale. \square

Prior to turning our attention to the infinitesimal generator, we first establish Ito differential formula for (4.1).

Theorem 4.1 (Ito Differential Formula). *Let $V \in \mathcal{C}[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}]$ be continuously differentiable in the first and second variables and twice continuously differentiable function in the fourth variable. Let x , y , N and γ be stochastic processes defined in (4.1), (2.14), (4.4) and (4.7), respectively. Moreover, processes N and ψ do not jump simultaneously P -almost surely. Then*

$$\begin{aligned}
 dV(s, y_{s-}, \theta_{s-}, x_{s-}) &= (\mathcal{L}V)(s, y_{s-}, \theta_{s-}, x_{s-})ds + \sigma(\theta_{s-})x_{s-} \frac{\partial V}{\partial x} dB_s \\
 &+ \int_{|z| \leq 1} [V(s, y_s, \theta_s, x_{s-} + x_{s-}G(z, \theta_s)) - V(y_s, \theta_s, x_{s-})]\bar{\psi}(\theta_s, dz, ds) \\
 &+ \int_{|z| > 1} [V(s, y_s, \theta_s, x_{s-} + x_{s-}H(z, \theta_s)) - V(s, y_s, \theta_s, x_{s-})]\bar{\psi}(\theta_s, dz, ds) \\
 &+ \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{s-}\}} \{V(s, y_s, j, x_{s-}e^z) \\
 &- V(s, y_{s-}, \theta_{s-}, x_{s-})\} \tilde{N}(ds, dz, \{(\theta_{s-}, j)\}), \tag{4.13}
 \end{aligned}$$

for $\theta_{s-} \in E$, where

$$\begin{aligned}
 \mathcal{L}V(s, y_{s-}, \theta_{s-}, x_{s-}) &= \frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(\theta_{s-})x_{s-} \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(\theta_{s-})x_{s-}^2 \frac{\partial^2 V}{\partial x^2} \\
 &+ \int_{|z| \leq 1} \left[V(s, y_{s-}, \theta_{s-}, x_{s-} + x_{s-}G(z, \theta_{s-})) \right. \\
 &- V(s, y_{s-}, \theta_{s-}, x_{s-}) - G(z, \theta_{s-})x_{s-} \frac{\partial V}{\partial x} \Big] \nu(\theta_{s-}, dz) \\
 &+ \int_{|z| > 1} [V(s, y_{s-}, \theta_{s-}, x_{s-} + x_{s-}H(z, \theta_{s-})) \\
 &- V(s, y_{s-}, \theta_{s-}, x_{s-})] \nu(\theta_{s-}, dz) \\
 &+ \int_{z \in \mathbb{R}} \sum_{j \neq \theta_{s-}} \lambda_{\theta_{s-}, j}(y_{s-}) [V(s, y_{s-}, j, x_{s-}e^z) \\
 &- V(s, y_{s-}, \theta_{s-}, x_{s-})] \bar{b}(z | \theta_{s-}, j) dz, \tag{4.14}
 \end{aligned}$$

$\theta_{s-} \in E$ and $\tilde{N} = N - \gamma$.

Proof. Let V be defined as in the theorem. Let $\{T_n\}_{n=1}^\infty$ be a sequence of semi-Markov jump times and $T_0 = 0$. For $t \in \mathbb{R}^+$, we can find an interval $[T_n, T_{n+1}]$ such that $T_n \leq t < t + \Delta t \leq T_{n+1}$ for some $n \in \mathbb{N}$. Let $\{J_j^n\}_{j=0}^{k_n} \subset [T_n, T_{n+1}]$ and $J_0^n = T_n$ be a finite sequence of jump times due to the Lévy jump process for $k_n \in \mathbb{N}$. We further note that the interval can be rewritten as

$$[T_n, T_{n+1}] = [T_n, T_{n+1}^-] \cup [T_{n+1}^-, T_{n+1}]. \tag{4.15}$$

We observe that $[J_j^n, J_{j+1}^{n-}] \cap [J_{j+1}^{n-}, J_{j+1}^n] = \emptyset$. In addition,

$$[T_n, T_{n+1}^-] = \bigcup_{j=0}^{k_n} ([J_j^n, J_{j+1}^{n-}] \cup [J_{j+1}^{n-}, J_{j+1}^n]). \quad (4.16)$$

It is known that the state dynamic process operating under the above stated conditions decomposes into three parts, namely, the continuous time, the Lévy jump time and the semi-Markov jump time. In fact, the solution process of (4.1)/(3.1) can be rewritten as

$$x_t = x_t^c + x_t^d + x_t^s, \quad (4.17)$$

where x_t^c , x_t^d and x_t^s are due to the presence of continuous process, Lévy process and semi-Markov process, respectively. We further observe that for $s \in [T_n, T_{n+1}]$, we have: $s = s^- + (s - s^-) = s^- + \Delta s$, where $\Delta s = s - s^-$, $s^- \neq s$. From Definitions 2.3, 4.1, we note that $y_s = y_{s^-}$ and $\theta_s = \theta_{s^-}$ for $s \in [T_n, T_{n+1}^-]$ and for $s = T_{n+1}$, $s \neq s^-$, $y_{s^-} \neq y_s$ and $\theta_{T_{n+1}^-} \neq \theta_{s^-}$. Moreover, there is a $j \in I(1, k_n - 1)$ such that $s \in [J_j^n, J_{j+1}^{n-}] \cup [J_{k_n}^n, T_{n+1}^-]$. We choose Δs so that $s + \Delta s \in [J_j^n, J_{j+1}^n]$. For these choices of s and $s + \Delta s$, we have

$$\begin{cases} y_{s+\Delta s} = y_{s^-} + \Delta s \\ \theta_{s+\Delta s} = \theta_{s^-} \\ x_{s+\Delta s} = x_{s^-} + \Delta x_s^s, \end{cases} \quad (4.18)$$

Furthermore,

$$\Delta x_s = \begin{cases} \Delta x_s^c, & \text{if } s \in [J_j^n, J_{j+1}^{n-}] \cup [J_{k_n}^n, T_{n+1}), \\ & \text{for } j \in I(0, k_n - 1) \text{ and } n \in I(0, \infty) \\ \Delta x_s^d, & \text{if } s = J_{j+1}^n, \text{ for } j \in I(1, k_n - 1) \\ \Delta x_s^s, & \text{if } s = T_{n+1} \end{cases} \quad (4.19)$$

(4.19) implies that for $s, \Delta s \in [J_j^n, J_{j+1}^{n-}]$, the change in state dynamic process is in the absence of the influence of Lévy jump process. On the other hand, for $s = J_j^n$ for each $j \in I(1, k_n)$, the dynamic process is interrupted by the presence of Lévy jumps. Finally, if $s = T_{n+1}$, then the dynamic system undergoes a structural change. Here the structural change is under the influence of the semi-Markov process. Therefore, there is no contribution of the continuous time dynamic process.

Based on the nature of the dynamic process operating under continuous time process, semi-Markov process and Lévy process, we compute the change in the auxiliary function V as

$$V(s + \Delta s, y_{s+\Delta s}, \theta_{s+\Delta s}, x_{s+\Delta s}) - V(s, y_s, \theta_s, x_s) \quad (4.20)$$

in the context of state dynamic model (3.1).

The computation of change in (4.20) depends on the computation of changes over the time domain of decomposition of $[T_n, T_{n+1}]$ for $n \in I(0, \infty)$. For computation

on $\bigcup_{j=0}^{k_n-1} [J_j^n, J_{j+1}^{n-}] \cup [J_{k_n}^n, T_{n+1}]$, we utilize the generalized mean value theorem. For this purpose, we pick $s, s + \Delta s \in [J_j^{n-}, J_{j+1}^n] \subset [T_n, T_n^-]$. From (4.18), the computation of (4.20) on the time domain $\bigcup_{j=0}^{k_n} [J_j^n, J_{j+1}^{n-}] \cup [J_{k_n}^n, T_{n+1}]$ regarding the continuous part of state dynamic (4.1)/(3.1) is as follows: The decomposition of three subsets of time domain $[T_n, T_{n+1}]$, namely, $\bigcup_{j=0}^{k_n} [J_j^n, J_{j+1}^{n-}] \cup [J_{k_n}^n, T_{n+1}]$, or $\bigcup_{j=1}^{k_n} [J_{j+1}^{n-}, J_{j+1}^n]$, or $[T_{n+}^-, T_{n+1}]$ for $n, k_n \in I(0, \infty)$.

$$\begin{aligned}
 & V(s + \Delta s, y_{s+\Delta s}, \theta_{s+\Delta s}, x_{s+\Delta s}) - V(s, y_s, \theta_s, x_s) \\
 &= \int_0^1 \left[\frac{\partial V}{\partial s}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \eta \Delta x_s) \Delta s \right. \\
 &\quad + \frac{\partial V}{\partial y}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \eta \Delta x_s) \Delta y_s \\
 &\quad \left. + \frac{\partial V}{\partial x}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \eta \Delta x_s) \Delta x_s \right] d\eta \\
 &= \frac{\partial V}{\partial s}(s, y_s, \theta_s, x_s) \Delta s + \frac{\partial V}{\partial y}(s, y_s, \theta_s, x_s) \Delta y_s + \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s) \Delta x_s \\
 &\quad + \int_0^1 \left[\frac{\partial V}{\partial x}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \eta \Delta x_s) - \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s) \right] \Delta x_s d\eta \\
 &\quad + \varepsilon_{s,y}(\Delta s), \tag{4.21}
 \end{aligned}$$

where

$$\begin{aligned}
 & \varepsilon_{s,y}(\Delta s) \\
 &= \int_0^1 \left[\frac{\partial V}{\partial s}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \eta \Delta x_s) - \frac{\partial V}{\partial s}(s, y_s, \theta_s, x_s) \right] \Delta s d\eta \\
 &\quad + \int_0^1 \left[\frac{\partial V}{\partial y}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \eta \Delta x_s) - \frac{\partial V}{\partial y}(s, y_s, \theta_s, x_s) \right] \Delta y_s d\eta. \tag{4.22}
 \end{aligned}$$

We again apply the generalized mean value theorem to the integrand in (4.21), and we obtain

$$\begin{aligned}
 & \frac{\partial V}{\partial x}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \eta \Delta x_s) - \frac{\partial V}{\partial x}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s) \\
 &= \frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s) \eta \Delta x_s + \int_0^1 \left[\frac{\partial^2 V}{\partial x^2}(s + \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \epsilon \eta \Delta x_s) \right. \\
 &\quad \left. - \frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s) \right] \Delta x_s \eta d\epsilon. \tag{4.23}
 \end{aligned}$$

From (4.23), the fourth term in (4.21) reduces to

$$\begin{aligned} & \int_0^1 \left[\frac{\partial V}{\partial x}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \eta \Delta x_s) - \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s) \right] \Delta x_s d\eta \\ &= \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s) (\Delta x_s)^2 + \varepsilon_x(\Delta s), \end{aligned} \quad (4.24)$$

where

$$\begin{aligned} \varepsilon_x(\Delta s) &= \int_0^1 \int_0^1 \left[\frac{\partial^2 V}{\partial x^2}(s + \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s + \epsilon \eta \Delta x_s) - \frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s) \right] \\ &\quad \times (\Delta x_s)^2 \eta d\eta d\epsilon \\ &\quad + \int_0^1 \left[\frac{\partial V}{\partial x}(s + \eta \Delta s, y_s + \eta \Delta y_s, \theta_s, x_s) - \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s) \right] d\eta \Delta x_s. \end{aligned}$$

From (4.21) and (4.24), we have

$$\begin{aligned} & V(s + \Delta s, y_s + \Delta y_s, \theta_{s+\Delta s}, x_{s+\Delta s}) - V(s, y_s, \theta_s, x_s) \\ &= \frac{\partial V}{\partial s}(s, y_s, \theta_s, x_s) \Delta s + \frac{\partial V}{\partial y}(s, y_s, \theta_s, x_s) \Delta y_s + \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s) \Delta x_s \\ &\quad + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s) (\Delta x_s)^2 + \varepsilon(\Delta s), \end{aligned} \quad (4.25)$$

where $\varepsilon(\Delta s) = \varepsilon_{s,y}(\Delta s) + \varepsilon_x(\Delta s)$. The expressions in (4.21), (4.23) and (4.24) are valid for all $s \in [J_j^n, J_{j+1}^{n-}] \cup [J_{k_n}^n, T_{n+1}^-]$ and for all $j \in I(1, k_n - 1)$ and hence they are valid on the intervals $[T_n, T_{n+1})$ for $n, k_n \in I(0, \infty)$.

Using Lévy integrals and a single jump value, we compute (4.20) under the influence of Lévy jump process. For this case, we first compute $V(J_{j+1}^n) - V(J_{j+1}^{n-})$, where $V(J_{j+1}^n) = V(s \wedge J_{j+1}^n, y_{s \wedge J_{j+1}^n}, \theta_{s \wedge J_{j+1}^n}, x_{s \wedge J_{j+1}^n} + x_{s \wedge J_{j+1}^n} G(z, \theta_{s \wedge J_{j+1}^n}))$ and $V(J_{j+1}^{n-}) = V(s, y_{s-}, \theta_{s-}, x_{s-})$.

We set and compute:

$$\begin{aligned} & V(s \wedge J_{j+1}^n) - V(s \wedge J_{j+1}^{n-}) \\ &= [V(s \wedge J_{j+1}^n, y_{s \wedge J_{j+1}^n}, \theta_{s \wedge J_{j+1}^n}, x_{s \wedge J_{j+1}^n} + x_{s \wedge J_{j+1}^n} G(z, \theta_{s \wedge J_{j+1}^n})) \\ &\quad - V(s, y_{s-}, \theta_{s-}, x_{s-})] \psi(\theta_{s-}, \Delta z, \Delta s) \\ &\quad + [V(s \wedge J_{j+1}^n, y_{s \wedge J_{j+1}^n}, \theta_{s \wedge J_{j+1}^n}, x_{s \wedge J_{j+1}^n} + x_{s \wedge J_{j+1}^n} H(z, \theta_{s \wedge J_{j+1}^n})) \\ &\quad - V(s, y_{s-}, \theta_{s-}, x_{s-})] \psi(\theta_{s-}, \Delta z, \Delta s). \end{aligned} \quad (4.26)$$

From (4.26), for any $s \in [J_{j+1}^{n-}, J_{j+1}^n]$, $j \in I(0, k_n - 1)$ and $n, k_n \in I(0, \infty)$ we have

$$\begin{aligned} & V(s \wedge J_{j+1}^n) - V(s \wedge J_{j+1}^{n-}) \\ &= \int_s^{s+\Delta s} \int_{|z| \leq 1} [V(s, y_s, \theta_s, x_{s-} + x_{s-} G(z, \theta_s)) - V(s, y_s, \theta_s, x_{s-})] \psi(\theta_s, dz, ds) \end{aligned}$$

$$\begin{aligned}
 & + \int_s^{s+\Delta s} \int_{|z|>1} [V(s, y_s, \theta_s, x_{s-} + x_{s-}H(z, \theta_s)) - V(s, y_s, \theta_s, x_{s-})] \\
 & \times \psi(\theta_s, dz, ds).
 \end{aligned} \tag{4.27}$$

The expression in (4.27) is over a subinterval $\bigcup_{j=0}^{k_n-1} [J_{j+1}^{n-}, J_{j+1}^n]$ of $[T_n, T_{n+1}]$. Finally, for $s \in [T_{n+1}^-, T_{n+1}]$, and imitating the above argument, we compute (4.20) under the presence of semi-Markov jump as follows:

$$\begin{aligned}
 & V(s \wedge T_{n+1}, y_{s \wedge T_{n+1}}, \theta_{s \wedge T_{n+1}}, x_{s \wedge T_{n+1}^-} + \Delta x_{s \wedge T_{n+1}^-}) \\
 & - V(s \wedge T_{n+1}^-, y_{s \wedge T_{n+1}^-}, \theta_{s \wedge T_{n+1}^-}, x_{s \wedge T_{n+1}^-}) \\
 & = V(s \wedge T_{n+1}, y_{s \wedge T_{n+1}^- + \Delta s}, \theta_{s \wedge T_{n+1}^- + \Delta s}, x_{s \wedge T_{n+1}^-} e^z) - V(s, y_{s-}, \theta_{s-}, x_{s-}) \\
 & = \int_s^{s+\Delta s} \int_{z \in \mathbb{R}} [V(u, y_{u-}, \theta_{u-}, x_{u-} e^z) \\
 & - V(u, y_{u-}, \theta_{u-}, x_{u-})] N(ds, dz, \{\theta_{T_{n+1}^-}, \theta_{T_{n+1}}\}),
 \end{aligned} \tag{4.28}$$

hence, adding and subtracting

$$\int_s^{s+\Delta s} \int_{z \in \mathbb{R}} [V(u, y_{u-}, \theta_{u-}, x_{u-} e^z) - V(u, y_{u-}, \theta_{u-}, x_{u-})] \gamma(ds, dz, \{\theta_{u-}, \theta_u\}),$$

we obtain

$$\begin{aligned}
 & V(s \wedge T_{n+1}, y_{s \wedge T_{n+1}}, \theta_{s \wedge T_{n+1}}, x_{s \wedge T_{n+1}^-} + \Delta x_{s \wedge T_{n+1}^-}) \\
 & - V(s \wedge T_{n+1}^-, y_{s \wedge T_{n+1}^-}, \theta_{s \wedge T_{n+1}^-}, x_{s \wedge T_{n+1}^-}) \\
 & = \int_s^{s+\Delta s} \int_{z \in \mathbb{R}} [V(u, y_{u-}, \theta_{u-}, x_{u-} e^z) - V(u, y_{u-}, \theta_{u-}, x_{u-})] \gamma(du, dz, \{\theta_{u-}, \theta_u\}) \\
 & + \int_s^{s+\Delta s} \int_{z \in \mathbb{R}} [V(u, y_{u-}, \theta_{u-}, x_{u-} e^z) - V(u, y_{u-}, \theta_{u-}, x_{u-})] \tilde{N}(du, dz, \{\theta_{u-}, \theta_u\}).
 \end{aligned} \tag{4.29}$$

This expression is on $[T_{n+1}^-, T_{n+1}]$ for $n \in I(0, \infty)$. From (4.25), (4.27) and (4.29), (4.20) reduces to

$$\begin{aligned}
 & V(s + \Delta s, y_{s+\Delta s}, \theta_{s+\Delta s}, x_{s+\Delta s}) - V(s, y_s, \theta_s, x_s) \\
 & = \frac{\partial V}{\partial s}(s, y_s, \theta_s, x_s) \Delta s + \frac{\partial V}{\partial y}(s, y_s, \theta_s, x_s) \Delta y_s + \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s) \Delta x_s \\
 & + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s) (\Delta x_s)^2
 \end{aligned}$$

$$\begin{aligned}
 & + \int_s^{s+\Delta s} \int_{|z| \leq 1} [V(u, y_u, \theta_u, x_{u-} + x_{u-} G(z, \theta_{u-})) \\
 & - V(u, y_{u-}, \theta_{u-}, x_{u-})] \psi(\theta_u, dz, du) \\
 & + \int_s^{s+\Delta s} \int_{|z| > 1} [V(u, y_u, \theta_u, x_{u-} + x_{u-} H(z, \theta_{u-})) \\
 & - V(u, y_{u-}, \theta_{u-}, x_{u-})] \psi(\theta_u, dz, du) \\
 & + \int_s^{s+\Delta s} \int_{z \in \mathbb{R}} \sum_{\theta_u \in E \setminus \{\theta_{u-}\}} [V(u, y_{u-}, \theta_{u-}, x_{u-} e^z) \\
 & - V(u, y_{u-}, \theta_{u-}, x_{u-})] \gamma(ds, dz, \{\theta_{u-}, \theta_u\}) \\
 & + \int_s^{s+\Delta s} \int_{z \in \mathbb{R}} [V(u, y_{u-}, \theta_{u-}, x_{u-} e^z) \\
 & - V(u, y_{u-}, \theta_{u-}, x_{u-})] \tilde{N}(ds, dz, \{\theta_{u-}, \theta_u\}) + \varepsilon(\Delta s). \tag{4.30}
 \end{aligned}$$

For small Δs , applying the concepts of stochastic differentials Applebaum (2009), adding and subtracting

$$\begin{aligned}
 & \int_s^{s+\Delta s} \int_{|z| > 1} [V(u, y_u, \theta_u, x_{u-} + x_{u-} H(z, \theta_{u-})) \\
 & - V(u, y_{u-}, \theta_{u-}, x_{u-})] \nu(\theta_{u-}, dz) du \quad \text{and} \\
 & \int_s^{s+\Delta s} \int_{|z| \leq 1} [V(u, y_u, \theta_u, x_{u-} + x_{u-} G(z, \theta_{u-})) \\
 & - V(u, y_{u-}, \theta_{u-}, x_{u-})] \nu(\theta_{u-}, dz) du,
 \end{aligned}$$

(4.30) reduces to

$$\begin{aligned}
 dV(s, y_{s-}, \theta_{s-}, x_{s-}) & = \mathcal{L}V(s, y_{s-}, \theta_{s-}, x_{s-}) ds + \sigma(\theta_{s-}) x_{s-} \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s) dB_s \\
 & + \int_{|z| > 1} [V(s, y_s, \theta_s, x_{s-} + x_{s-} H(z, \theta_{s-}) x_{s-}) \\
 & - V(s, y_{s-}, \theta_{s-}, x_{s-})] \bar{\psi}(\theta_s, dz, ds) \\
 & + \int_{|z| \leq 1} [V(s, y_s, \theta_s, x_{s-} + x_{s-} G(z, \theta_{s-}) x_{s-}) \\
 & - V(s, y_{s-}, \theta_{s-}, x_{s-})] \bar{\psi}(\theta_s, dz, ds) \\
 & + \int_{z \in \mathbb{R}} [V(s, y_{s-}, \theta_{s-}, x_{s-} e^z) \\
 & - V(s, y_{s-}, \theta_{s-}, x_{s-})] \tilde{N}(ds, dz \times \{\theta_{s-}, \theta_s\}). \tag{4.31}
 \end{aligned}$$

This establishes Ito differential formula (4.13) for Lévy type stochastic differential equation under semi-Markovian structural perturbations. Here \mathcal{L} in (4.14) is the linear differential operator relative to (4.1). \square

In the following, based on Theorem 4.1, we present a concept of infinitesimal generator and a few results as special cases.

Definition 4.3. For the function V defined in Theorem 4.1 and using (4.30), an infinitesimal generator of (4.1) is defined by

$$\begin{aligned} & \lim_{\Delta t \rightarrow 0} \left[\frac{1}{\Delta t} E[V(t + \Delta t, y_{t+\Delta t}, \theta_{t+\Delta t}, x_{t+\Delta t}) \right. \\ & \quad \left. - V(t, y_t, \theta_t, x_t) \mid y_t = y_{t-}, \theta_t = \theta_{t-}, x_t = x_{t-} \right] \\ & = \mathcal{A}V(t, y_{t-}, \theta_{t-}, x_{t-}), \quad \text{for } \theta_{t-} \in E. \end{aligned} \quad (4.32)$$

Moreover, a one parameter family of semi-group is generated by

$$\frac{\partial V}{\partial t}(t, y_{t-}, \theta_{t-}, x_{t-}) = \mathcal{A}V(t, y_{t-}, \theta_{t-}, x_{t-}), \quad (4.33)$$

where $\mathcal{A} = \mathcal{L}$ in (4.13) and $\frac{\partial V}{\partial t}(t, y_t, \theta_t, x_t)$ is the conditional partial derivative defined by the left-hand side of (4.32).

We present special cases of the developed infinitesimal generator in Definition 4.3.

Remark 4.4. From Remark 3.2, the infinitesimal generator \mathcal{A} defined in (4.32) extends the earlier work in a systematic way. In fact, this generator includes the infinitesimal generator influenced by finite state Markov chain (Elliott *et al.* 2005, Chourdakis 2005). Moreover, it also includes the generator influenced by a finite state semi-Markov process (Ghosh & Goswami 2009, Hunt & Hahn 2010, Swishchuk & Islam 2011). If H and G are replaced by $e^G - 1$ and $e^H - 1$, then \mathcal{A} in (4.32) in the context of (4.14) is

$$\begin{aligned} & \mathcal{A}V(s, y_{s-}, \theta_{s-}, \theta_s, x_{s-}) \\ & = \frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(\theta_{s-})x_{s-} \frac{\partial V}{\partial x} + \frac{1}{2}x_{s-}^2 \sigma^2(\theta_{s-}) \frac{\partial^2 V}{\partial x^2} \\ & \quad + \int_{|z| \leq 1} \left[V(s, y_{s-}, \theta_{s-}, \theta_s, x_{s-} + x_{s-}[e^{G(z, \theta_{s-})} - 1]) \right. \\ & \quad \left. - V(s, y_{s-}, \theta_{s-}, \theta_s, x_{s-}) - x_{s-}[e^{G(z, \theta_{s-})} - 1] \frac{\partial V}{\partial x} \right] \nu(\theta_{s-}, dz) \\ & \quad + \int_{|z| > 1} [V(s, y_{s-}, \theta_{s-}, \theta_s, x_{s-} + x_{s-}[e^{H(z, \theta_{s-})} - 1]) \end{aligned}$$

$$\begin{aligned}
 & -V(s, y_{s-}, \theta_{s-}, \theta_s, x_{s-})] \nu(\theta_{s-}, dz) \\
 & + \int_{z \in \mathbb{R}} \sum_{\theta_s \in E, \theta_s \neq \theta_{s-}} \lambda_{\theta_{s-}, \theta_s}(y_s) [V(s, y_{s-}, \theta_s, x_{s-} e^z) \\
 & - V(s, y_{s-}, \theta_{s-}, x_{s-})] \bar{b}(z | \theta_{s-}, \theta_s) dz.
 \end{aligned}$$

A few notes on the nature of the infinitesimal operator.

Remark 4.5. We further remark that the infinitesimal generator defined in (4.32) can be rewritten in a $m \times m$ matrix form. In fact the partial differential equations in (4.33) are a system of partial differential equations of dimension m . More precisely, (4.33) is a linear system of partial differential equations with variable coefficients.

Remark 4.6. For $V(t, y_t, \theta_t, x_t) = x_t$, the conclusion of Theorem 4.1 reduces to

$$\begin{aligned}
 dx_t &= \mathcal{L}V(t, y_{t-}, \theta_{t-}, x_{t-})dt + \sigma(\theta_{t-})dB_t + \int_{|z| \leq 1} x_{t-} G(z, \theta_{t-}) \bar{\psi}(\theta_{t-}, dz, dt) \\
 &+ \int_{|z| > 1} x_{t-} H(z, \theta_{t-}) \bar{\psi}(\theta_{t-}, dz, dt) \\
 &+ \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} [x_{t-} (e^z - 1) \bar{N}(dt, dz, \{(\theta_{t-}, j)\})], \tag{4.34}
 \end{aligned}$$

where

$$\begin{aligned}
 & \mathcal{L}V(t, y_{t-}, \theta_{t-}, x_{t-}) \\
 &= x_{t-} \left[\mu(\theta_{t-}) + \int_{|z| > 1} H(z, \theta_{t-}) \nu(\theta_{t-}, dz) \right. \\
 & \quad \left. + \int_{|z| > 1} \sum_{j \in E \setminus \{\theta_{t-}\}} [(e^z - 1) \lambda_{\theta_{t-}, j}(y_{t-})] \bar{b}(z | \theta_{t-}, j) dz \right]. \tag{4.35}
 \end{aligned}$$

$$dx_t = x_{t-} dM_t^\theta, \tag{4.36}$$

with

$$\begin{aligned}
 dM_t^\theta &= \mu(\theta_{t-})dt + \int_{|z| \leq 1} G(z, \theta_{t-}) \bar{\psi}(\theta_{t-}, dz, dt) \\
 &+ \int_{|z| > 1} H(z, \theta_{t-}) \bar{\psi}(\theta_{t-}, dz, dt) \\
 &+ \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} [(e^z - 1) \bar{N}(dt, dz, \{(\theta_{t-}, j)\})]
 \end{aligned}$$

$$\begin{aligned}
& + \int_{|z|>1} \sum_{j \in E \setminus \{\theta_{t-}\}} [(e^z - 1) \lambda_{\theta_{t-}, j}(y_{t-})] \bar{b}(z | \theta_{t-}, j) dz dt \\
& + \int_{|z|>1} H(\theta_{t-}, z) \nu(\theta_{t-}, dz) dt \\
& = dL_t^\theta + \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} [(e^z - 1) N(dt, dz, \{(\theta_{t-}, j)\})], \tag{4.37}
\end{aligned}$$

where L^θ is defined in (3.2). Furthermore, we note that the solution process determined by (4.36) has another solution representation of (4.1) in the framework of Remark 4.2. In fact, the closed form solution representation of (4.36) is as follows:

$$\begin{aligned}
x_t = x_0 \exp & \left[\int_0^t \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{s-}\}} [z N(ds, dz, \{(\theta_{s-}, j)\})] \right. \\
& + \int_0^t \mu(\theta_{s-}) ds - \frac{1}{2} \int_0^t \sigma^2(\theta_{s-}) ds \\
& + \int_0^t \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) ds \\
& + \int_0^t \sigma(\theta_{s-}) dB_s + \int_0^t \int_{|z| \leq 1} \ln(1 + G(z, \theta_{s-})) \bar{\psi}(\theta_{s-}, dz, ds) \\
& \left. + \int_0^t \int_{|z| > 1} \ln(1 + H(z, \theta_{s-})) \psi(\theta_{s-}, dz, ds) \right]. \tag{4.38}
\end{aligned}$$

In the following section, we utilize the infinitesimal generator of the exponential semi-Markov Lévy switching process to find a closed form expression of the characteristic function of the $\ln(x_t)$ from (4.1).

5. Characteristic Function

In this section, we derive a closed form expression for the conditional characteristic function

$$\Psi(u, t, y, j, x) = E[e^{iu \ln(x_t)} | y_0 = y, \theta_0 = j, x_0 = x] \tag{5.1}$$

of the log price process

$$\ln(x_t) = \sum_{p=1}^{n(t)} \ln(\beta_p) + L_t^\theta, \tag{5.2}$$

where β_n , L^θ in Definition 4.1 and x_t is the closed form solution process of (4.1) in the context of (3.25).

Lemma 5.1. *Let L_t^θ , x , y and γ be defined in (3.2), (4.1), (2.14) and (4.7), respectively. A closed form expression for the conditional characteristic vector function of*

$\ln(x)$ is

$$\Psi(u, t, y, x) = \exp[iu \ln(x)] \exp \left[\int_y^{t+y} M(u, s) ds \right] \cdot \mathbf{1}, \quad (5.3)$$

where $i = \sqrt{-1}$; $\Psi(u, t, y, x)$ is an m -dimensional column vector with k th component $\Psi(u, t, y, k, x)$, for $k \in E$; $\mathbf{1}$ is $m \times 1$ vector with components ones, and $M(u, y) = (M_{pq}(u, y))_{m \times m}$ is an $m \times m$ matrix defined by

$$M_{q,p}(u, y) = \begin{cases} iu\mu(q) - \frac{1}{2}\sigma^2(q)u^2 + \int_{|z| \leq 1} [e^{iuG(z,q)} - 1 - iuG(z,q)]\nu(q, dz) \\ \quad + \int_{|z| > 1} [e^{iuH(z,q)} - 1]\nu(q, dz) + \lambda_{q,q}(y), & \text{if } p = q \\ \lambda_{q,p}(y) \int_{z \in \mathbb{R}} e^{iuz} \bar{b}(z | q, p) dz, & \text{otherwise,} \end{cases}$$

and it is assumed to satisfy the Lie bracket-type condition

$$[M(u, y_1), M(u, y_2)] = 0, \quad \forall y_1, y_2 \in \mathbb{R}^+. \quad (5.4)$$

Proof. From (5.3), first we observe that

$$\Psi(u, t, y, x) = \exp[iuL_t^\theta] \exp \left[\int_y^{t+y} M(u, s) ds \right] \cdot \mathbf{1}. \quad (5.5)$$

We note that $\Psi(u, t, y, \theta_t, x)$ possesses all smoothness properties of V defined in Theorem 4.1. Therefore, following the argument used in the proof of Theorem 4.1, Definition 4.3 and Remark 4.5, we conclude that Ψ is in the domain of the infinitesimal generator of the process $(y_t, \theta_t, iuL_t^\theta)_{t \in [0, T]}$. Moreover, it satisfies the following system of linear partial differential equation:

$$\frac{\partial \Psi(u, t, y, k, x)}{\partial t} = \mathcal{A}\Psi(u, t, y, k, x), \quad \text{for } k \in E, \quad (5.6)$$

where \mathcal{A} is the operator defined in Definition 4.3. From Remark 4.4 with $\mu(\theta_{s-})$ replaced by

$$\mu(\theta_{s-}) + \frac{1}{2}\sigma(\theta_{s-}) + \int_{|z| \leq 1} [e^{G(z, \theta_{s-})} - G(Z, \theta_{s-}) - 1]\nu(\theta_{s-}, dz)$$

and for Ψ defined in (5.1), we have

$$\begin{aligned} & \mathcal{A}\Psi(u, s, y_{s-}, \theta_{s-}, x) \\ &= \frac{\partial \Psi(u, s, y, \theta_{s-}, x)}{\partial s} + \frac{\partial \Psi(u, s, y, \theta_{s-}, x)}{\partial y} \\ & \quad + iu \left[\mu(\theta_s) + \frac{1}{2}\sigma^2(\theta_s) \right] x_{s-} \frac{\partial \Psi(u, s, y, \theta_{s-}, x)}{\partial x} \end{aligned}$$

$$\begin{aligned}
& + \int_{|z| \leq 1} \left[\Psi(u, s, y, \theta_{s-}, x + x[e^{iuG(z, \theta_{s-})} - 1]) \right. \\
& \quad \left. - \Psi(u, s, y, \theta_{s-}, x) - iuxG(z, \theta_{s-}) \frac{\partial \Psi(u, s, y, \theta_{s-}, x)}{\partial x} \right] \nu(\theta_{s-}, dz) \\
& + \int_{|z| > 1} [\Psi(u, s, y, \theta_{s-}, x + x[e^{iuH(z, \theta_{s-})} - 1]) \\
& \quad - \Psi(u, s, y, \theta_{s-}, x)] \nu(\theta_{s-}, dz) - \frac{1}{2} x_s^2 u^2 \sigma^2(\theta_{s-}) \frac{\partial^2 \Psi(u, s, y, \theta_{s-}, x)}{\partial x^2} \\
& + \int_{z \in \mathbb{R}} \sum_{\theta_s \in E \setminus \{\theta_{s-}\}} \Psi(u, s, y, \theta_s, x e^{iuz}) \bar{b}(z | \theta_{s-}, \theta_s) \lambda_{\theta_{s-}, \theta_s}(y) dz \\
& - \Psi(u, s, y, \theta_{s-}, x) \lambda_{\theta_{s-}, \theta_{s-}}(y) \quad \text{for } \theta_{s-} \in E. \tag{5.7}
\end{aligned}$$

Now, we assume that $\Psi(u, t, y, k, x) = \exp[iu \ln(x)] h(u, t, y, k)$, where $h(u, t, y, k)$ is the k th component of an unknown m -dimensional vector function $\mathbf{h}(u, t, y) = [h(u, t, y, 1), \dots, h(u, t, y, m)]^\top$. From this, (5.6) reduces to the following system of partial differential equations:

$$\begin{aligned}
\frac{\partial h(u, t, y, k)}{\partial t} &= \frac{\partial h(u, t, y, k)}{\partial y} + h(u, t, y, k) \left[iu \left[\mu(k) + \frac{1}{2} \sigma^2(k) \right] \right. \\
& \quad + \frac{1}{2} \sigma^2(k) [-iu - u^2] + \int_{|z| \leq 1} [e^{iuG(z, k)} - 1 - iuG(z, k)] \nu(k, dz) \\
& \quad + \left. \int_{|z| > 1} [e^{iuH(z, k)} - 1] \nu(k, dz) + \lambda_{k, k}(y) \right] \\
& \quad + \int_{z \in \mathbb{R}} \sum_{j \neq k} \lambda_{k, j}(y) h(u, t, y, j)(y) e^{iuz} \bar{b}(z | k, j) dz \\
&= \frac{\partial h(u, t, y, k)}{\partial y} + h(u, t, y, k) \left[iu \mu(k) - \frac{1}{2} \sigma^2(k) u^2 \right. \\
& \quad + \int_{|z| \leq 1} [e^{iuG(z, k)} - 1 - iuG(z, k)] \nu(k, dz) \\
& \quad + \left. \int_{|z| > 1} [e^{iuH(z, k)} - 1] \nu(k, dz) + \lambda_{k, k}(y) \right] \\
& \quad + \int_{z \in \mathbb{R}} \sum_{j \neq k} \lambda_{k, j}(y) h(u, t, y, j)(y) e^{iuz} \bar{b}(z | k, j) dz. \tag{5.8}
\end{aligned}$$

As stated in Remark 4.5, the coefficients of h are defined by the elements of \mathcal{A} associated with $\Psi(u, t, y, x)$, in particular, the $m \times m$ matrix $M(u, y) = (M_{k, j}(u, y))_{m \times m}$

defined in (5.4). From the definition of $\mathbf{h}(u, t, y)$, (5.8) reduces to

$$\frac{\partial \mathbf{h}(u, t, y)}{\partial t} = \frac{\partial \mathbf{h}(u, t, y)}{\partial y} + M(u, y)\mathbf{h}(u, t, y), \quad \mathbf{h}(u, 0, y) = \mathbf{1} = \underbrace{(1, \dots, 1)^\top}_{m \text{ ones}}. \quad (5.9)$$

Using the method of characteristics, the system of partial differential equations (5.9) can be solved. In this case, the characteristic curves are determined by $\frac{dy}{dt} = \pm 1$. Solving these differential equations, we obtain

$$\eta = t - y \quad \text{and} \quad \zeta = t + y. \quad (5.10)$$

We use the above change of variable to define the transforms \tilde{h} and \tilde{M} from h and M , respectively, as functions of (η, ζ)

$$\begin{cases} \tilde{\mathbf{h}}(u, \eta, \zeta) = \mathbf{h}\left(u, \frac{\eta + \zeta}{2}, \frac{-\eta + \zeta}{2}\right) \\ \tilde{M}(u, \eta, \zeta) = M\left(u, \frac{-\eta + \zeta}{2}\right). \end{cases} \quad (5.11)$$

From (5.11), the initial value problem (5.9) reduces to the ODE

$$\frac{\partial \tilde{\mathbf{h}}(u, \eta, \zeta)}{\partial \eta} = \frac{1}{2} \tilde{M}(u, \eta, \zeta) \tilde{\mathbf{h}}(u, \eta, \zeta), \quad \tilde{\mathbf{h}}(u, -y, y) = \mathbf{1}. \quad (5.12)$$

Under condition (5.4), the general solution of the linear homogeneous ODE with time varying coefficients is Magnus (1954)

$$\tilde{\mathbf{h}}(u, \eta, \zeta) = \exp\left[\frac{1}{2} \int_0^\eta \tilde{M}(u, \kappa, \zeta) d\kappa\right] \cdot g(\zeta), \quad (5.13)$$

where g is an arbitrary m -dimensional vector function. Using the initial condition in (5.12), g is determined by

$$g(\zeta) = \exp\left[\frac{1}{2} \int_{-\zeta}^0 \tilde{M}(u, \kappa, \zeta) d\kappa\right] \mathbf{1}, \quad \forall \zeta \in [0, T].$$

This together with (5.13), yields the solution of the initial value problem (5.12) as

$$\tilde{\mathbf{h}}(u, \eta, \zeta) = \exp\left[\frac{1}{2} \int_{-\zeta}^\eta \tilde{M}(u, \kappa, \zeta) d\kappa\right] \mathbf{1}.$$

Using the inverse of the transformation defined in (5.10), the solution of the original initial value problem (5.9) becomes

$$\begin{aligned} \mathbf{h}(u, t, y) &= \exp\left[\frac{1}{2} \int_{-t-y}^{t-y} M(u, \frac{-\kappa + t + y}{2}) d\kappa\right] \mathbf{1} \\ &= \exp\left[\int_y^{t+y} M(u, s) ds\right] \mathbf{1}. \end{aligned}$$

This establishes the conditional characteristic function for the log prices described by (4.1). \square

Remark 5.1. We note that the closed form exponential expression (5.3) holds only under condition (5.4). This is due to the fact that system of ode in (5.12) has time varying coefficients. Assuming the matrix $(M_{i,j})_{m \times m}$ has continuous entries with respect to y , the system of differential equations with time varying coefficients (5.12) has a fundamental matrix $\Phi(u, t, y)$. Therefore, the characteristic function in (5.1) is described by

$$\Psi(u, t, y, x) = \exp[iu \ln(x)][\Phi(u, t, y)\Phi(u, 0, y)^{-1}]\mathbf{1}. \quad (5.14)$$

The characteristic function $\Psi(u, t, y, x)$ has a closed form expression if Φ has a closed form expression. Theorem 5.1 corresponds to the particular case where $\Psi(u, t, y, x) = \exp[\int_y^{t+y} M(u, s)ds]$.

In the following, we present a consequence of Lemma 5.1 that extends the characteristic function of sojourn time of finite state Markov processes Momeya (2012) and Elliott & Osakwe (2006).

Corollary 5.1. *We denote $O_s^t(k)$ the time spent by the semi-Markov process $(\theta_t)_{t \in [0, T]}$ in its state k in the time interval $[s, t]$, $\forall s, t \in [0, T]$, with $s \leq t$, and $O_s^t = [O_s^t(1), O_s^t(2), \dots, O_s^t(m)]^\top$ denotes the m -dimensional occupation time vector of θ . If $a = (a_1, a_2, \dots, a_m)^\top$ is an $m \times 1$ vector of constant real numbers, then*

$$E[e^{iu \langle J_s^t, a \rangle} | y_s = y, \theta_s = k] = \left\langle \left[\exp \left[\int_y^{t+y} M(u, s)ds \right] \right] \cdot e_k, \mathbf{1} \right\rangle,$$

where

$$M_{p,q}(u, y) = \begin{cases} iua_q + \lambda_{q,q}(y), & \text{if } p = q \\ \lambda_{p,q}(y), & \text{otherwise.} \end{cases} \quad (5.15)$$

Proof. Let $(a_t)_{t \in [0, T]}$ denote a stochastic process with $a_t = a_j$ whenever $\theta_{n(t)} = j$.

$$\begin{aligned} \exp[iu \langle O_s^t, a \rangle] &= \exp \left[iu \sum_{k \in E} a_k O_s^t(k) \right], \quad \text{for } i = \sqrt{-1} \\ &= \exp \left[iu \int_s^t a_v - dv \right], \quad (\text{as } a \text{ is piecewise constant}) \\ &= \exp \left[iu \int_s^t a_v - dv \right] \\ &= e^{iu \int_s^t dL_v^\theta}, \quad \text{with } L_v^\theta = a_v, \quad \forall v \in [0, T]. \end{aligned}$$

The characteristic function of the semi-Markov occupation times in the time interval $[s, t]$ becomes

$$\begin{aligned} E[e^{iu \langle J_s^t, a \rangle} | y_s = y, \theta_s = k] \\ &= E[e^{iu \int_s^t dL_v^\theta} | y_s = y, \theta_s = k] \\ &= E[e^{iu L_{t-s}^\theta} | y_0 = y, \theta_0 = k], \end{aligned}$$

since the couple (θ, y) is homogeneous. Applying Lemma 5.1 with $\beta_n = 1 \forall n \in I(0, \infty)$ proves the result. Corollary 5.1 is a direct extension of the results in Hainaut & Colwell (2014), Buffington & Elliott (2002) and Momeya (2012). \square

6. Change of Measure and Pricing Kernels

In this section, we introduce the conditional minimum equivalent martingale measuring the semi-Markov jump risk and the Lévy risk. In addition, we also develop an unconditional minimum entropy martingale measure and the Esscher transform Siu & Yang (2009) measuring all three risks, namely, Lévy risk, semi-Markov jump risk and regime switching risk. Prior to the development of these concepts and results, we utilize the closed form solution representation of (3.1) to shed a light on the martingale property of the solution process of the Lévy type stochastic linear differential equations. For this purpose, let x_t be the solution process of (3.1) and assume that it is a $(\mathbb{H}_t \vee \bar{\mathbb{L}}_t)$ -martingale, that is, for $s \leq t$, $E[x_t - x_s | \mathbb{H}_s \vee \bar{\mathbb{L}}_s] = 0$. This is represented by the following illustrations.

Illustration 6.1.

- (1) From $E[x_t - x_s | \mathbb{H}_s \vee \bar{\mathbb{L}}_s] = 0$ and (3.25), it is obvious that the solution process x_t of (3.1)

$$x_t = x_0 \exp \left[\int_0^t \mu(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s + \int_0^t \int_{|z| \leq 1} G(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) + \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) \right] \quad (6.1)$$

is a martingale if and only if

$$\begin{aligned} \mu(\theta_{t-}) + \frac{1}{2} \sigma^2(\theta_{t-}) + \int_{|z| \leq 1} [e^{G(z, \theta_{t-})} - G(z, \theta_{t-}) - 1] \nu(\theta_{t-}, dz) \\ + \int_{|z| > 1} [e^{H(z, \theta_{t-})} - 1] \nu(\theta_{t-}, dz) = 0, \quad \forall \theta_{t-} \in E. \end{aligned} \quad (6.2)$$

- (2) Furthermore if L_t^θ in (3.2) is replaced by M_t^θ

$$\begin{aligned} dM_t^\theta = \sigma(\theta_{t-}) dB_t + \int_{|z| \leq 1} G(z, \theta_{t-}) \bar{\psi}(\theta_{t-}, dz, dt) \\ + \int_{|z| > 1} H(z, \theta_{t-}) \bar{\psi}(\theta_{t-}, dz, dt), \end{aligned} \quad (6.3)$$

then the solution process of (3.1) in (3.18) is indeed a martingale and is represented by

$$\begin{aligned} x_t = x_0 \exp \left[-\frac{1}{2} \int_0^t \sigma^2(\theta_{s-}) ds + \int_0^t \sigma(\theta_{s-}) dB_s \right. \\ \left. + \int_0^t \int_{|z| > 1} [\ln(H(z, \theta_{s-}) + 1) - H(z, \theta_{s-})] \nu(\theta_{s-}, dz) ds \right] \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t \int_{|z| \leq 1} \ln(1 + G(z, \theta_{s-})) \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_0^t \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) ds \\
 & + \int_0^t \int_{|z| > 1} \ln(1 + H(z, \theta_{s-})) \bar{\psi}(\theta_{s-}, dz, ds) \Big]. \tag{6.4}
 \end{aligned}$$

- (3) Replacing $H(z, \theta_s)$, $G(z, \theta_s)$ and L_s^θ in (3.2) by $e^{H(z, \theta_s)} - 1$, $e^{G(z, \theta_s)} - 1$ and M^θ in (6.3), respectively, the solution of the (IVP) (3.1) in (3.20) is a martingale if and only if

$$\begin{aligned}
 x_t = x_0 \exp & \left[\int_0^t \sigma(\theta_{s-}) dB_s + \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \nu(\theta_{s-}, dz) ds \right. \\
 & - \frac{1}{2} \int_0^t \sigma^2(\theta_{s-}) ds + \int_0^t \int_{|z| \leq 1} [G(z, \theta_{s-}) - e^{G(z, \theta_{s-})} + 1] \nu(\theta_{s-}, dz) ds \\
 & \left. + \int_0^t \int_{|z| \leq 1} G(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) + \int_0^t \int_{|z| > 1} H(z, \theta_{s-}) \bar{\psi}(\theta_{s-}, dz, ds) \right]. \tag{6.5}
 \end{aligned}$$

- (4) $V(t, y_{t-}, \theta_{t-}, x_{t-})$ in (4.13) is a martingale if and only if $\mathcal{L}V(t, y_{t-}, \theta_{t-}, x_{t-})$ is identically equal to zero. In particular, from (4.34), the solution process (4.1) is a local martingale if and only if $\mathcal{L}V(t, y_{t-}, \theta_{t-}, x_{t-})$ in (4.35) is identically zero that is $\mu(\theta_{t-}) + \int_{|z| > 1} H(z, \theta_{t-}) \nu(\theta_{t-}, dz) + \int_{|z| > 1} \sum_{j \in E \setminus \{\theta_{t-}\}} [(e^z - 1) \lambda_{\theta_{t-}, j}(y_{t-})] \bar{b}(z | \theta_{t-}, j) dz = 0$.

We introduce and recall a few notations necessary for presenting the next lemma.

Remark 6.1. We denote Φ_t a positive $(P, (\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale process with initial value $\Phi_0 = 1$. In fact for $x_0 = 1$, any one of the solution processes in Illustration 6.1 can be represented by Φ_t , that is the fundamental solution process of linear Lévy-type stochastic differential equations. Moreover, Φ_t is called a density process of a probability measure \bar{P} with respect to a given probability measure P .

Based on a Girsanov theorem for Jacod & Shiryaev (1987) and point Brémaud (1981) processes, we present a Girsanov-type theorem for stochastic hybrid process described by (4.1). We highlight the effects of change of measures on both time and state domains of decomposition with respect to $(L_t^\theta)_{t \in [0, T]}$, $(\beta_n)_{n \geq 0}$ and $(\theta_t)_{t \in [0, T]}$. $(T_n)_{n \geq 0}$ are the jump times in Definition 2.1.

Lemma 6.1 (Girsanov-type Theorem). *Let η and Y be piecewise deterministic stochastic processes defined on $[0, T] \times \mathbb{R}$ and $[0, T] \times \mathbb{R} \times \mathbb{R}$ into \mathbb{R} , respectively.*

$\xi = (\xi_{i,j}(s, z))_{m \times m}$ is a $\mathbb{R}^{m \times m}$ -valued and $\bar{\mathbb{H}}_t$ -predictable process defined on $[0, T] \times \mathbb{R}$ into \mathbb{R} . Let us consider the process M_t^θ defined by

$$dM_t^\theta = \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} (e^z - 1) N(dt, dz, \{(\theta_{t-}, j)\}) + dL_t^\theta, \quad (6.6)$$

where L_t^θ is defined in (3.2). Furthermore, we make the following assumptions

(H1) $\xi = (\xi_{i,j}(s, z))_{m \times m}$, $(\lambda_{i,j}(y_s))_{m \times m}$ defined in (2.9), η and Y satisfy the following conditions:

$$\left\{ \begin{array}{l} \int_0^t \int_{z \in \mathbb{R}} \xi_{i,j}(z, s) \lambda_{i,j}(y_{s-}) \bar{b}(z | i, j) dz ds < \infty \quad Y \geq 0 \\ \int_0^t \eta(s, \theta_{s-}) \mu(\theta_{s-}) ds < \infty \\ \int_{z \in \mathbb{R}} [G(z, \theta_{s-}) 1_{|z| \leq 1} + H(z, \theta_{s-}) 1_{|z| > 1}] [Y(\theta_{s-}, z, s) - 1] \nu(\theta_{s-}, dz) < \infty \\ \int_{z \in \mathbb{R}} [Y(\theta_{s-}, z, s) - 1]^2 \nu(\theta_{s-}, dz) < \infty. \end{array} \right. \quad (6.7)$$

(H2) let Z_t be the solution process of the following linear SDE,

$$\begin{aligned} dZ_t = Z_{t-} & \left[\eta(t, \theta_t) \sigma(\theta_t) dB_s + \int_{z \in \mathbb{R}} (Y(\theta_t, z, t) - 1) \bar{\psi}(\theta_t, dz, dt) \right. \\ & \left. + \sum_{(i,j) \in E^2} \int_{z \in \mathbb{R}} [\xi_{i,j}(t, z) - 1] \bar{N}(dt, dz, \{(i, j)\}) \right] Z_0 = 1, \end{aligned} \quad (6.8)$$

where $\bar{N} = N - \gamma$ defined in (4.14) and Z_t has the closed form representation

$$\begin{aligned} Z_t = \exp & \left[- \int_0^t \frac{1}{2} \eta(s, \theta_{s-})^2 \sigma(\theta_{s-})^2 ds + \int_0^t \eta(s, \theta_{s-}) \sigma(\theta_{s-}) dB_s \right. \\ & + \int_0^t \int_{z \in \mathbb{R}} (Y(\theta_{s-}, z, s) - 1) \bar{\psi}(\theta_{s-}, dz, ds) \\ & + \int_0^t \int_{z \in \mathbb{R}} [\ln(Y(\theta_{s-}, z, s)) - (Y(\theta_{s-}, z, s) - 1)] \psi(\theta_{s-}, dz, ds) \Big] \\ & \times \prod_{(i,j) \in E^2} \exp \left[\int_0^t (1 - \xi_{i,j}(s, z)) \bar{b}(z | i, j) \lambda_{i,j}(y_{s-}) dz ds \right. \\ & \left. + \int_0^t \int_{z \in \mathbb{R}} \ln(\xi_{i,j}(s, z)) N(ds, dz, \{(i, j)\}) \right]. \end{aligned} \quad (6.9)$$

Therefore, from Remark 6.1 and under a local equivalent probability measure \bar{P} with density process Z_t with respect to P , the following hold:

- (1) $B_t^{\bar{P}} = -\int_0^t \eta(s, \theta_{s-}) \sigma(\theta_{s-}) ds + B_t$ is a Brownian motion for each $\theta_{t-} \in E$,
- (2) $\nu^{\bar{P}}(\theta_{t-}, \cdot) = Y(\theta_{t-}, t, \cdot) \nu(\theta_{t-}, \cdot)$ P -almost surely,
- (3) $\gamma^{\bar{P}}(dz, \{(i, j)\}) = \xi_{i,j}(t, z) \bar{b}(z | i, j) \lambda_{i,j}(y_t) dz$ P -almost surely,
- (4) M_t^θ defined in (6.6) can be expressed as follows:

$$\begin{aligned}
 dM_t^\theta = & \left[\mu(\theta_{t-}) + \sigma^2(\theta_{t-}) \eta(t, \theta_{t-}) + \int_{|z| \leq 1} G(z, \theta_{t-}) (Y(\theta_{t-}, z, t) - 1) \nu(\theta_{t-}, dz) \right. \\
 & + \int_{|z| > 1} H(z, \theta_{t-}) Y(\theta_{t-}, z, t) \nu(\theta_{t-}, dz) \\
 & \left. + \sum_{j \in E \setminus \{\theta_{t-}\}} \int_{z \in \mathbb{R}} [e^z - 1] \gamma^{\bar{P}}(dz, \{\theta_{t-}, j\}) \right] dt + \sigma(\theta_{t-}) dB^{\bar{P}} \\
 & + \int_{|z| \leq 1} G(z, \theta_{t-}) \bar{\psi}^{\bar{P}}(\theta_{t-}, dz, dt) + \int_{|z| > 1} H(z, \theta_{t-}) \bar{\psi}^{\bar{P}}(\theta_{t-}, dz, dt) \\
 & + \sum_{j \in E \setminus \{\theta_{t-}\}} \int_{z \in \mathbb{R}} [e^z - 1] [N(dt, dz, \{\theta_{t-}, j\}) - \gamma^{\bar{P}}(dz, \{\theta_{t-}, j\}) dt]. \quad (6.10)
 \end{aligned}$$

Proof. From (6.8), we note that $E[Z_t - Z_s | \bar{\mathbb{H}}_s \vee \mathbb{L}_s] = 0, \forall s, t \in [0, T]$ with $s \leq t$. Hence, Z_t is a local martingale. From the initial condition $Z_0 = 1$ in (6.8) and Illustration 6.1, Z_t is a density process of \bar{P} . Moreover, $\bar{P}(A) = \int_A Z_t(w) dP(w)$, for $A \in \bar{\mathbb{H}}_t \vee \mathbb{L}_t$. Consequently, \bar{P} is a local equivalent probability measure with density Z_t relative to P . From the definition of $B^{\bar{P}}$ in 1, it is obvious that it is a Brownian motion with mean $\int_0^t \eta(s, \theta_{s-}) \sigma(\theta_{s-}) ds$ and variance t . It remains to show that $B^{\bar{P}}$ is a local martingale with respect to \bar{P} . For this purpose, we use (6.8) and apply Ito formula for the product $ZB^{\bar{P}}$ Ladde & Ladde (2013) and we have,

$$\begin{aligned}
 d(Z_t B_t^{\bar{P}}) &= Z_t dB^{\bar{P}} + B_t^{\bar{P}} dZ_t + dZ_t dB_t^{\bar{P}} \\
 &= Z_t dB + B_t^{\bar{P}} dZ_t \\
 &= Z_t [1 + \eta(t, \theta_{t-}) \sigma(\theta_{t-}) B_t^{\bar{P}}] dB_t \\
 &\quad + Z_t B_t^{\bar{P}} \left[\int_{z \in \mathbb{R}} (Y(\theta_t, z, t) - 1) \bar{\psi}(\theta_t, dz, dt) \right. \\
 &\quad \left. + \sum_{(i,j) \in E^2} \int_{z \in \mathbb{R}} [\xi_{i,j}(t, z) - 1] \bar{N}(dt, dz, \{(i, j)\}) \right].
 \end{aligned}$$

From this, we conclude that $B^{\bar{P}}$ is a \bar{P} -continuous local martingale with quadratic variation t . From Lévy characterization of Brownian motions, $B^{\bar{P}}$ is a

\bar{P} -standard Brownian motion. This establishes 1. We now prove that $\nu^{\bar{P}}(\theta_{t-}, dz) = Y(\theta_{t-}, t, z)\nu(\theta_{t-}, dz)$ is the \bar{P} -intensity measure of $\psi(\theta_{t-}, \cdot, \cdot)$. Knowing that \bar{P} and P are equivalent, following the argument Applebaum (2009), we define the conditional characteristic function for the Poisson process $\psi(\theta_{t-}, \cdot, \cdot)$ relative to the probability measure \bar{P} as follows:

$$E^{\bar{P}} \left[\exp \left[iu \int_0^t \int_{z \in \mathbb{R}} \psi(\theta_{s-}, dz, ds) \right] \middle| \mathbb{H}_T \right] = \exp \left[\int_0^t \int_{z \in \mathbb{R}} [(e^{iu} - 1)] \nu^{\bar{P}}(\theta_s, dz) \right], \quad (6.11)$$

where $\nu^{\bar{P}}$ is an intensity measure of ψ with respect to \bar{P} . Using the closed form expression of the density process (6.9), the characteristic function in (6.11) is also computed as follows:

$$\begin{aligned} & E^{\bar{P}} \left[\exp \left[iu \int_0^t \int_{z \in \mathbb{R}} \psi(\theta_{s-}, dz, ds) \right] \middle| \mathbb{H}_T \right] \\ &= E \left[Z_t \exp \left[iu \int_0^t \int_{z \in \mathbb{R}} \psi(\theta_{s-}, dz, ds) \right] \middle| \mathbb{H}_T \right] \\ &= E \left[\exp \left[\int_0^t \int_{z \in \mathbb{R}} (Y(\theta_{s-}, z, s) - 1) \bar{\psi}(\theta_{s-}, dz, ds) \right. \right. \\ &\quad \left. \left. + \int_0^t \int_{z \in \mathbb{R}} [\ln(Y(\theta_{s-}, z, s)) - Y(\theta_{s-}, z, s) + 1 + iu] \psi(\theta_{s-}, dz, ds) \right] \middle| \mathbb{H}_T \right] \\ &= \exp \left[\int_0^t \int_{z \in \mathbb{R}} (1 - Y(\theta_{s-}, z, s)) \nu(\theta_{s-}, dz) ds \right] \\ &\quad \times E \left[\exp \left[\int_0^t \int_{z \in \mathbb{R}} (Y(\theta_{s-}, z, s) - 1) \psi(\theta_{s-}, dz, ds) \right. \right. \\ &\quad \left. \left. + \int_{z \in \mathbb{R}} [\ln(Y(\theta_{s-}, z, s)) - (Y(\theta_{s-}, z, s) - 1) + iu] \psi(\theta_s, dz, ds) \right] \middle| \mathbb{H}_T \right] \\ &= \exp \left[- \int_0^t \int_{z \in \mathbb{R}} (Y(\theta_{s-}, z, s) - 1) \nu(\theta_{s-}, dz) ds \right] \\ &\quad \times E \left[\exp \left[\int_0^t \int_{z \in \mathbb{R}} [\ln(Y(\theta_{s-}, z, s)) + iu] \psi(\theta_{s-}, dz, ds) \right] \middle| \mathbb{H}_T \right]. \quad (6.12) \end{aligned}$$

We note that $\int_0^t \int_{z \in \mathbb{R}} [\ln(Y(\theta_{s-}, z, s)) + iu] \psi(\theta_{s-}, dz, ds)$ is a compound Poisson process. From Applebaum (2009), (6.12) becomes

$$\begin{aligned} & E^{\bar{P}} \left[\exp \left[iu \int_0^t \int_{z \in \mathbb{R}} \psi(\theta_{s-}, dz, ds) \right] \middle| \mathbb{H}_T \right] \\ &= \exp \left[- \int_0^t \int_{z \in \mathbb{R}} (Y(\theta_{s-}, z, s) - 1) \nu(\theta_{s-}, dz) ds \right] \end{aligned}$$

$$\begin{aligned}
&= \exp \left[\int_0^t \int_{z \in \mathbb{R}} [e^{\ln(Y(\theta_s, z, s)) + iu} - 1] \nu(\theta_s, dz) ds \right] \\
&= \exp \left[\int_0^t \int_{z \in \mathbb{R}} [Y(\theta_{s-}, z, s) e^{iu} - Y(\theta_{s-}, z, s)] \nu(\theta_{s-}, dz) \right] \\
&= \exp \left[\int_0^t \int_{z \in \mathbb{R}} [(e^{iu} - 1)] Y(\theta_{s-}, z, s) \nu(\theta_{s-}, dz) \right]. \tag{6.13}
\end{aligned}$$

From (6.11) and (6.13), it is obvious that the intensity of Lévy jump poisson measure ψ , with respect to \bar{P} , is $\bar{\nu} = Y\nu$ \bar{P} — almost surely. Based on the proof of 2, the proof of 3 can be reformulated, analogously. The verification of (6.10) follows from algebraic computations. \square

Remark 6.2. We recall that under the historical probability measure P , $(p_{i,j})_{(i,j) \in E^2}$ in (2.5), $F(|i, j)$ in Lemma 2.1 and $\bar{b}(|i, j)$ in (4.3) are the transition probability matrix of the embedded Markov chain, the sojourn time distribution and the log jump density, respectively. We denote $(p_{i,j}^{\bar{P}})_{(i,j) \in E^2}$, $F^{\bar{P}}(|i, j)$ and $\bar{b}^{\bar{P}}(|i, j)$ the transition probability matrix, the conditional cumulative distribution of sojourn times and the density of the log of jump due to the semi-Markov process from state i to state j at jump time T_{n-1} , under the probability measure \bar{P} . Using these notions and part 3 of Lemma 6.1, we have

$$\begin{aligned}
\bar{b}^{\bar{P}}(z | i, j) \lambda_{i,j}^{\bar{P}}(y_s) &= \xi_{i,j}(s, z) \bar{b}(z | i, j) \lambda_{i,j}(y_s), \quad \text{with} \\
\lambda_{i,j}^{\bar{P}}(y_s) &= p_{i,j}^{\bar{P}} \frac{f^{\bar{P}}(y_s | i, j)}{1 - \sum_{k \neq i} p_{i,j}^{\bar{P}} F^{\bar{P}}(y_s | i, k)}. \tag{6.14}
\end{aligned}$$

We further remark that \bar{P} is a risk neutral measure, if the process $L_t^\theta - \int_0^t r(s) ds$ is a local martingale with respect to \bar{P} , whenever the drift coefficient satisfies the condition:

$$\begin{aligned}
&\mu(\theta_{t-}) - r(t) + \sigma^2(\theta_{t-}) \eta(t, \theta_{t-}) + \int_{|z| \leq 1} G(z, \theta_{t-}) (Y(\theta_{t-}, z, t) - 1) \nu(\theta_{t-}, dz) \\
&+ \int_{|z| > 1} H(z, \theta_{t-}) Y(\theta_{t-}, z, t) \nu(\theta_{t-}, dz) \\
&+ \sum_{j \in E \setminus \{\theta_{t-}\}} \int_{z \in \mathbb{R}} [e^z - 1] \gamma^{\bar{P}}(dz, \{(\theta_{t-}, j)\}) = 0. \tag{6.15}
\end{aligned}$$

Given the 2-variate process $(\eta(t, \theta_t), Y(\theta_t, z, t))$ in (6.8), one can freely choose ξ . Hence, for each choice of ξ , one gets a distinct risk neutral measure. Furthermore, by the application of the first and the second fundamental theorem of asset pricing Back & Pliska (1991), the market under consideration is arbitrage free and incomplete.

Following arguments in Elliott *et al.* (2005), Momeya (2012) and Miyahara (1999), we define two particular equivalent martingale measures, namely the conditional and the unconditional minimum entropy martingale measure, respectively.

6.1. Conditional minimum entropy martingale measure

We define the conditional minimum entropy martingale measure (CMEMM) pricing Lévy and semi-Markov jump risks. In the absence of risk associated with regime changes, a pricing kernel is computed through a random Esscher transform. Without loss in generality, we assume that investors always know past and future market regimes. Based on the idea in Miyahara *et al.* (2001), we define the process R^θ as follows:

$$R_t^\theta = \int_0^t \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{s-}\}} [(e^z - 1)N(ds, dz, \{(\theta_{s-}, j)\})] + L_t^\theta, \quad (6.16)$$

where L_t^θ , $n(t)$ and β_k are defined in (3.2), (2.3) and (4.1), respectively. Picking a locally bounded process $(\alpha_t)_{t \in [0, T]}$, we modify the process defined in (6.16) as

$$\begin{aligned} \int_0^t \alpha_s dR_s^\theta &= \int_0^t \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{s-}\}} \alpha_{s-} [(e^z - 1)N(ds, dz, \{(\theta_{s-}, j)\})] \\ &\quad + \int_0^t \alpha_{s-} dL_s^\theta. \end{aligned} \quad (6.17)$$

In the following, we utilize the modified process (6.17) to formulate a dynamic process for the asset process.

Definition 6.1. Let α be a locally bounded process. We assume that $E[e^{\int_0^t \alpha_{s-} dR_s^\theta} | \mathbb{H}_T] < \infty$, $\forall t \in [0, T]$. We define the stochastic processes Z^α and $k(s, z, ds, dz)$ as follows:

$$Z_t^\alpha = \frac{e^{\int_0^t \alpha_{s-} dR_s^\theta}}{E[e^{\int_0^t \alpha_{s-} dR_s^\theta} | \mathbb{H}_T]}, \quad \forall t \geq 0 \quad (6.18)$$

and

$$k(s, z, ds, dz) = \sum_{j \in E \setminus \{\theta_{s-}\}} \alpha_{s-} (e^z - 1) N(ds, dz, \{(\theta_{s-}, j)\}), \quad \forall s \geq 0, \quad z \in \mathbb{R}. \quad (6.19)$$

The stochastic process defined in (6.18) is called an Esscher transformation with Esscher parameters $(\alpha_s)_{s \in [0, T]}$.

We first establish preliminary results useful for finding a necessary and sufficient condition under which the probability measure P^α with density relative to P defined by the Esscher transform in (6.18) is an equivalent martingale measure relative to the asset price process x_t described by (4.1).

Lemma 6.2. *Under Definition 4.1, Remark 4.1 and the Esscher parameter $(\alpha_s)_{s \in [0, T]}$ in Definition 6.1, a stochastic process x_t^α exists and satisfies the following properties.*

(1)

$$\begin{aligned} E[x_t^\alpha \mid \mathbb{H}_T] \\ = \prod_{i=0}^{n(t)} E \left[\exp \left[\alpha_i \int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz) \right] \middle| \theta_i, \theta_{i+1} \right] e^{\int_{T_i}^{T_{i+1}} f_i(s) ds}, \end{aligned} \quad (6.20)$$

where

$$\begin{aligned} f_i(s) &= \alpha(i) \mu(\theta_i) + \frac{1}{2} \sigma^2(\theta_i) \alpha(i) \\ &\quad + \int_{|z| \leq 1} [e^{\alpha(i)G(z, \theta_i)} - 1 - \alpha(i)G(z, \theta_i)] \nu(\theta_i, dz) \\ &\quad + \int_{|z| > 1} [e^{\alpha(i)H(z, \theta_i)} - 1] \nu(\theta_i, dz), \end{aligned} \quad (6.21)$$

for $i \in I(0, \infty)$, $s \in [0, T]$.

(2)

$$\frac{x_t^\alpha}{E[x_t^\alpha \mid \mathbb{H}_T]} = \prod_{i=0}^{n(t)} \frac{\exp[\alpha_i \int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] e^{\int_{T_i}^{T_{i+1}} \alpha_s - dM_s^{\theta_i}}}{E[\exp[\alpha_i \int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \mid \theta_i, \theta_{i+1}]}, \quad (6.22)$$

where $\alpha_s - dM_s^{\theta_i} = \alpha_s - dL_s^{\theta_i} - f_i(s)ds$.

(3) $E[\frac{x_t^\alpha}{E[x_t^\alpha \mid \mathbb{H}_T]} \mid \mathbb{H}_T] = 1$,

(4) $Z_t^\alpha = \frac{x_t^\alpha}{E[x_t^\alpha \mid \mathbb{H}_T]}$ is a $(P, \mathbb{H}_T \vee \bar{\mathbb{L}})$ -local martingale.

(5) If P^α is a risk neutral measure with respect to Z_t^α , then under P^α we have:

(a) $B_t^{P^\alpha} = B_t - \int_0^t \alpha_s - \sigma(\theta_{s-}) ds$, is a P^α -standard Brownian motion process.

(b) $\nu^{P^\alpha}(\theta_{s-}, dz) = e^{[H(z, \theta_{s-})1_{(|z| > 1)} + G(z, \theta_{s-})1_{(|z| \leq 1)}]} \nu(\theta_{s-}, dz)$, is a P^α -predictable compensator of the Poisson random measure $\psi(j, \cdot)$ for all $j \in E$.

(c) The density of the n th jump coefficient β_n is

$$\frac{\exp[\int_{T_n}^{T_{n+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_n}^{T_{n+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \mid \theta_n, \theta_{n+1}]}.$$

Proof. From Definition 4.1, $0 = T_0 \leq T_1 \leq T_2 \leq \dots \leq T_{n-1}$ are the regime switching times caused by the semi-Markov process prior to t . For notational convenience, we denote $\theta_{-1} = \theta_0$. Under the assumption of the lemma, the solution process of (4.1) in the context of (4.2) and the simple return process (6.17) exist and it is represented as

$$x_t^\alpha = \prod_{i=0}^{n(t)} \exp \left[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz) \right] e^{\int_{T_i}^{T_{i+1}} \alpha_s - dL_s^{\theta_i}},$$

with $\beta_0 = x_0 = 1$. For $t \in [T_n, T_{n+1}]$, from the independence of Lévy and semi-Markov processes, we have

$$\begin{aligned} E[x_t^\alpha \mid \mathbb{H}_T] &= \prod_{i=0}^{n-1} \left[E \left[\exp \left[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz) \right] \middle| \theta_{i-1}, \theta_i \right] \right. \\ &\quad \left. \times E[e^{\int_{T_i}^{T_{i+1}} \alpha_s - dL_s^{\theta_i}} e^{\int_{T_i}^t \alpha_s - dL_s^{\theta_i}} \mid \mathbb{H}_T] \right]. \end{aligned} \quad (6.23)$$

This, together with an application of the Lévy Kintchine formula Øksendal & Sulem (2005) yields

$$\begin{aligned} E[x_t^\alpha \mid \mathbb{H}_T] &= \prod_{i=0}^{n(t)} E \left[\exp \left[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz) \right] \middle| \theta_{i-1}, \theta_i \right] \\ &\quad \times \left[\exp \int_{T_i}^{T_{i+1}} \left[\alpha_s - \mu(\theta_i) + \frac{1}{2} \sigma^2(\theta_i) \alpha_s^2 - \right. \right. \\ &\quad \left. \left. + \int_{|z| \leq 1} [e^{\alpha_s - G(z, \theta_i)} - 1 - \alpha_s - G(z, \theta_i)] \nu(\theta_i, dz) \right. \right. \\ &\quad \left. \left. + \int_{|z| > 1} [e^{\alpha_s - H(z, \theta_i)} - 1] \nu(\theta_i, dz) \right] ds \right]. \end{aligned}$$

This completes the proof of (1). For the proof of (2), we consider

$$\frac{x_t^\alpha}{E[x^\alpha \mid \mathbb{H}_T]}. \quad (6.24)$$

From (6.21) and (6.24), we obtain

$$\begin{aligned} &\frac{x_t^\alpha}{E[x^\alpha \mid \mathbb{H}_T]} \\ &= \frac{\prod_{i=0}^{n(t)} \exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \exp[\int_{T_i}^{T_{i+1}} \alpha_s - dL_s^{\theta_i}]}{\prod_{i=0}^{n(t)} E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \mid \theta_j, \theta_{j+1}] \exp[\int_{T_j}^{T_{j+1}} f_j(s) ds]} \\ &= \prod_{i=0}^{n(t)} \frac{\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \exp[\int_{T_i}^{T_{i+1}} [\alpha_s - dL_s^{\theta_j} - f_j(s) ds]]}{E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \mid \theta_{j-1}, \theta_j]}. \end{aligned} \quad (6.25)$$

From (6.3), (6.17), (6.21) and (6.25), we observe that $\alpha_s - dL_s^{\theta_j} - f_j(s) ds$ has a form similar to (6.3), that is

$$\alpha_s - dL_s^{\theta_j} - f_j(s) ds = \alpha_s - dM_t^{\theta_j}, \quad (6.26)$$

with coefficients G and H replaced by $e^G - 1$ and $e^H - 1$, respectively, hence establishing (2). Using (1), (6.25) and (6.26), we further remark that

$$\begin{aligned}
 & E \left[\frac{x_t^\alpha}{E[x_t^\alpha | \mathbb{H}_T]} \middle| \mathbb{H}_T \right] \\
 &= E \left[\prod_{i=0}^{n(t)} \frac{\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \exp[\int_{T_i}^{T_{i+1}} \alpha_s - dM_s^{\theta_i}]}{E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_i, \theta_{i+1}]} \middle| \mathbb{H}_T \right] \\
 &= \prod_{i=0}^{n(t)} \frac{E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_{i-1}, \theta_i]}{E[E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_i, \theta_{i+1}] | \theta_{i-1}, \theta_i]} \\
 &\quad \times E \left[\exp \left[\int_{T_i}^{T_{i+1}} [\alpha_s - dL_s^{\theta_i} - f_i(s) ds] \right] \middle| \mathbb{H}_T \right] \\
 &= \prod_{i=0}^{n(t)} 1 = 1, \quad \text{for } t \in [0, T],
 \end{aligned}$$

which establishes (3). For the proof of (4) we consider

$$\begin{aligned}
 \frac{\frac{x_t^\alpha}{E[x_t^\alpha | \mathbb{H}_T]}}{\frac{x_s^\alpha}{E[x_s^\alpha | \mathbb{H}_T]}} &= \prod_{i=n(s)+1}^{n(t)} \frac{\exp[\alpha_i \int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_{i-1}, \theta_i]} \\
 &\quad \times \exp \left[\int_{T_i}^{T_{i+1}} [\alpha_s - dM_s^{\theta_i}] \right].
 \end{aligned}$$

The conditional expectation with respect to $\mathbb{H}_T \vee \bar{\mathbb{L}}_s$ yields

$$E \left[\frac{x_t^\alpha}{E[x_t^\alpha | \mathbb{H}_T]} \middle| \mathbb{H}_T \vee \bar{\mathbb{L}}_s \right] = \frac{x_s^\alpha}{E[x_s^\alpha | \mathbb{H}_T]}.$$

This proves (4). Moreover, from (1), (4) and (6.22), Z^α is a probability density process of a probability measure P^α with respect to P . The proof of statements in (5(a)) and (5(b)) of (5) follow by imitating the proofs of (1) and (2) of Lemma 6.1. We only establish (5(c)). For $B \subset \mathbb{B}_k$ and $t \in [T_k, T_{k+1}]$. In fact

$$\begin{aligned}
 E^{P^\alpha}[1_B] &= E[1_B Z_t^{P^\alpha}] \\
 &= E[E[1_B Z_t^{P^\alpha} | \mathbb{H}_T]], \\
 &= E \left[E \left[1_B \prod_{i=0}^{n(t)} \frac{\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_{i-1}, \theta_i]} \middle| \mathbb{H}_T \right] \right]
 \end{aligned}$$

$$\begin{aligned}
 &= E \left[1_B \frac{\exp[\int_{T_k}^{T_{k+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_k}^{T_{k+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_{k-1}, \theta_k]} \right] \\
 &\quad \times \prod_{i=1, i \neq k}^{n(t)} E \left[\frac{\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_{i-1}, \theta_i]} \middle| \theta_{i-1}, \theta_i \right] \\
 &= E \left[1_B \frac{\exp[\int_{T_k}^{T_{k+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_k}^{T_{k+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_{k-1}, \theta_k]} \right].
 \end{aligned}$$

Hence, $\forall B \in \mathbb{B}_k$, $E^{P^\alpha}[1_B] = E[1_B \frac{\exp[\int_{T_k}^{T_{k+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_k}^{T_{k+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_{k-1}, \theta_k]}]$. From Radon Nikodym theorem Jacod & Shiryaev (1987), the density of β_k under P^α is $\frac{\exp[\int_{T_k}^{T_{k+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_k}^{T_{k+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] | \theta_{k-1}, \theta_k]}$. This completes the proof of the lemma. \square

In the following lemma, we provide a sufficient condition for the price process to be a $(P^\alpha, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale. The result obtained will be used to derived the martingale condition on the discounted price process.

Lemma 6.3. *In addition to assumptions of Lemma 6.2, we assume that $\int_{|z|>1} (H(z, \theta_s) + 1) e^{\alpha(j)H(z, \theta_s)} \nu(j, dz) < \infty, \forall j \in E$. Then the following results hold:*

(1) x in (4.1) is a $(P^\alpha, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale measure provided that

$$\begin{cases} \mu(\theta_n) + \alpha_t \sigma^2(\theta_n) + \int_{|z| \leq 1} G(z, \theta_n) [e^{\alpha_t G(z, \theta_n)} - 1] \nu(\theta_n, dz) \\ \quad + \int_{|z| > 1} H(z, \theta_n) e^{\alpha_t H(z, \theta_n)} \nu(\theta_n, dz) = 0, \\ E_{P^\alpha}[\beta_n | \theta_{n-1}, \theta_n] = 1, \quad \forall t \in (T_n, T_{n+1}), \quad \forall n \in I(0, \infty). \end{cases} \quad (6.27)$$

(2) The discounted price process $\tilde{x}_t = e^{\int_0^t r_s ds} x_t$, is a $(P^\alpha, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale if:

$$\begin{cases} \mu(\theta_n) + \alpha_t \sigma^2(\theta_n) + \int_{|z| \leq 1} [G(z, \theta_n) e^{\alpha_t G(z, \theta_n)} - G(z, \theta_n)] \nu(\theta_n, dz) \\ \quad + \int_{|z| > 1} [e^{\alpha_t H(z, \theta_n)} - 1] \nu(\theta_n, dz) = r_t, \\ E_{P^\alpha}[\beta_n | \theta_{n-1}, \theta_n] = 1, \quad \forall t \in (T_n, T_{n+1}), \quad \forall n \in I(0, \infty). \end{cases}$$

(3) Let α^* and P^{α^*} be a solution process of equation (6.27) and the probability measure associated with the density process Z^{α^*} , respectively. Under P^{α^*} , the

process R_t^θ in (6.16) could be expressed as follows:

$$\begin{aligned} dR_t^\theta &= \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} \alpha_{t-}(e^z - 1) N(dt, dz, \{(\theta_{t-}, j)\}) + r_{t-} dt + \sigma(\theta_{t-}) dB^{P^{\alpha^*}} \\ &\quad + \int_{|z| \leq 1} G(z, \theta_{t-}) [\psi(\theta_{t-}, dt, dz) - \nu^{P^{\alpha^*}}(\theta_{t-}, dz) dt] \\ &\quad + \int_{|z| > 1} H(z, \theta_{t-}) [\psi(\theta_{t-}, dt, dz) - \nu^{P^{\alpha^*}}(\theta_{t-}, dz) dt], \end{aligned}$$

with

$$E_{P^{\alpha^*}} [\beta_n \mid \theta_{n-1}, \theta_n] = 1.$$

Proof. From Radon Nikodym theorem, x_t is a $(P^\alpha, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale if and only if $x_t Z_t^\alpha$ is a $(P, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale. From (3.18) and (4.1)

$$x_t Z_t^\alpha = x_s Z_s^\alpha \prod_{i=n(s)+1}^{n(t)} \left[\frac{\beta_i \exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] e^{\int_{T_i}^{T_{i+1}} [\alpha_{s-} dM_s^{\theta_i} + d\bar{L}_s^{\theta_i}]} }{E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \mid \theta_{i-1}, \theta_i \mid \theta_{i-1}, \theta_i]} \right] \quad (6.28)$$

with \bar{L}_t^θ defined as follows:

$$\begin{aligned} d\bar{L}_s^\theta &= \left[\mu(\theta_{s-}) - \frac{1}{2} \sigma^2(\theta_{s-}) + \int_{|z| \leq 1} [\ln(1 + G(z, \theta_{s-})) - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) \right] ds \\ &\quad + \sigma(\theta_{s-}) B_s + \int_{|z| \leq 1} \ln(1 + G(z, \theta_{s-})) \bar{\psi}(\theta_{s-}, dz, ds) \\ &\quad + \int_{|z| > 1} \ln(1 + H(z, \theta_{s-})) \psi(\theta_{s-}, dz, ds). \end{aligned} \quad (6.29)$$

From (6.3), (6.28) and (6.29), we have

$$\begin{aligned} &\alpha_{s-} dM_s^{\theta_i} + d\bar{L}_s^\theta \\ &= \left[\mu(\theta_{s-}) - \frac{1}{2} \alpha_{s-}^2 \sigma^2(\theta_{s-}) - \frac{1}{2} \sigma^2(\theta_{s-}) ds - \int_{|z| > 1} [e^{\alpha_{s-} H(z, \theta_{s-})} - 1] \nu(\theta_{s-}, dz) \right. \\ &\quad + \int_{|z| \leq 1} [\alpha_{s-} G(z, \theta_{s-}) - e^{\alpha_{s-} G(z, \theta_{s-})} + 1 \\ &\quad \left. + \ln(G(z, \theta_{s-}) + 1) - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) ds \right] + (\alpha_{s-} + 1) \sigma(\theta_{s-}) dB_s \\ &\quad + \int_{|z| \leq 1} [\alpha_{s-} G(z, \theta_{s-}) + \ln(G(z, \theta_{s-}) + 1)] \bar{\psi}(\theta_{s-}, dz, ds) \\ &\quad + \int_{|z| > 1} [\alpha_{s-} H(z, \theta_{s-}) + \ln(H(z, \theta_{s-}) + 1)] \psi(\theta_{s-}, dz, ds). \end{aligned} \quad (6.30)$$

From (3.18), $d[e^{\int_0^t d(\alpha_s - M_s^\theta + \bar{L}_s^\theta)}] = e^{\int_0^t d(\alpha_s - M_s^\theta + \bar{L}_s^\theta)} dL_t^\star$ with

$$\begin{aligned}
 dL_s^\star = & \left[\mu(\theta_{s-}) + \alpha_{s-} \sigma^2(\theta_{s-}) \right. \\
 & + \int_{|z| \leq 1} [G(z, \theta_{s-}) e^{\alpha_{s-} G(z, \theta_{s-})} - G(z, \theta_{s-})] \nu(\theta_{s-}, dz) ds \\
 & \left. - \int_{|z| > 1} H(z, \theta_{s-}) e^{\alpha_{s-} H(z, \theta_{s-})} \nu(\theta_{s-}, dz) \right] ds + \sigma(\theta_{s-})(\alpha_{s-} + 1) dB_s \\
 & + \int_{|z| \leq 1} [(G(z, \theta_{s-}) + 1) e^{\alpha_{s-} G(z, \theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds) \\
 & + \int_{|z| > 1} [(H(z, \theta_{s-}) + 1) e^{\alpha_{s-} H(z, \theta_{s-})} - 1] \bar{\psi}(\theta_{s-}, dz, ds). \tag{6.31}
 \end{aligned}$$

We now derive conditions under which $x_t Z_t^\alpha$ is a $(P, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale process. $x_t Z_t^\alpha$ is a $(P, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale process if and only if

$$E[x_t Z_t^\alpha \mid \mathbb{H}_T \vee \bar{\mathbb{L}}_s] = x_s Z_s^\alpha, \quad \forall s, t \in [0, T]. \tag{6.32}$$

Applying Lemma 4.1 to $V(s, y_s, \theta_s, Z_s x_s) = x_t Z_t$ and replacing G, H, σ, μ and β_i by

$$\begin{aligned}
 & (G(z, \theta_s) + 1) e^{\alpha_s G(z, \theta_s)} - 1, \\
 & (H(z, \theta_s) + 1) e^{\alpha_s H(z, \theta_s)} - 1, \\
 & (\alpha_s + 1) \sigma_s, \\
 & \mu(\theta_s) + \alpha_s \sigma^2(\theta_s) + \int_{|z| \leq 1} [G(z, \theta_s) e^{\alpha_s G(z, \theta_s)} - G(z, \theta_s)] \nu(\theta_s, dz) \\
 & - \int_{|z| > 1} [e^{\alpha_s H(z, \theta_s)} - 1] \nu(\theta_s, dz) \quad \text{and} \\
 & \frac{\beta_i \exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_i}^{T_{i+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)] \mid \theta_{i-1}, \theta_i]},
 \end{aligned}$$

respectively, we obtain

$$\begin{aligned}
 x_t Z_t^\alpha - x_s Z_s^\alpha = & \int_s^t x_u - Z_u^\alpha \left[\mu(\theta_{u-}) + \alpha_{u-} \sigma^2(\theta_{u-}) \right. \\
 & + \int_{|z| \leq 1} [G(z, \theta_{u-}) e^{\alpha_{u-} G(z, \theta_{u-})} - G(z, \theta_{u-})] \nu(\theta_{u-}, dz) \\
 & \left. - \int_{|z| > 1} [e^{\alpha_{u-} H(z, \theta_{u-})} - 1] \nu(\theta_{u-}, dz) \right] du
 \end{aligned}$$

$$\begin{aligned}
& + \int_{|z|>1} x_{u-} Z_{u-}^{\alpha} [(H(u, \theta_{u-}) + 1)e^{\alpha_{u-} H(z, \theta_{u-})} - 1] \nu(\theta_{u-}, dz) \Big] du \\
& + \int_s^t \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{u-}\}} x_{u-} Z_{u-}^{\alpha} (e^z - 1) N(du, dz, \{\theta_{u-}, j\}) \\
& + \underbrace{\text{sum of martingale terms.}}
\end{aligned}$$

Taking the conditional expectation, we obtain

$$\begin{aligned}
& E[x_t Z_t^{\alpha} - x_s Z_s^{\alpha} \mid \mathbb{H}_T \vee \bar{\mathbb{L}}_s] \\
& = \int_s^t E[x_{u-} Z_{u-}^{\alpha} \mid \mathbb{H}_T \vee \bar{\mathbb{L}}_s] \left[\mu(\theta_{u-}) + \alpha_{u-} \sigma^2(\theta_{u-}) \right. \\
& \quad + \int_{|z| \leq 1} [G(z, \theta_{u-}) e^{\alpha_{u-} G(z, \theta_{u-})} - G(z, \theta_{u-})] \nu(\theta_{u-}, dz) \\
& \quad \left. - \int_{|z|>1} H(z, \theta_{u-}) e^{\alpha_{u-} H(z, \theta_{u-})} \nu(\theta_{u-}, dz) \right] du \\
& \quad + E \left[\int_s^t \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{u-}\}} \alpha_{u-} x_{u-} Z_{u-}^{\alpha} (e^z - 1) N(du, dz, \{(\theta_{u-}, j)\}) \Bigg| \mathbb{H}_T \vee \bar{\mathbb{L}}_s \right] \\
& = 0, \quad \forall s, t \in [0, T], \tag{6.33}
\end{aligned}$$

for any s, t and for small Δs $s, t = s + \Delta s \in (T_n, T_{n+1})$ for some $n \in I(1, \infty)$. This together with (6.33) yields

$$\begin{aligned}
& \mu(\theta_s) + \alpha_s \sigma^2(\theta_s) + \int_{|z| \leq 1} [G(z, \theta_s) e^{\alpha_s G(z, \theta_s)} - G(z, \theta_s)] \nu(\theta_s, dz) \\
& + \int_{|z|>1} H(s, \theta_s) e^{\alpha_s H(z, \theta_s)} \nu(\theta_s, dz) \Delta s = 0, \quad \forall s \in (T_n, T_{n+1}), \quad n \in I(1, \infty). \tag{6.34}
\end{aligned}$$

Lastly, we assume $[s, t] = [T_n, T_{n+1}]$. When Δs is small. There is one regime change $[s, t]$ at $t = T_n$. Using (6.34) and (6.33) becomes

$$\begin{aligned}
& E \left[\int_{T_n}^{T_{n+1}} \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{u-}\}} \alpha_{u-} [x_{u-} Z_{u-}^{\alpha}] \mid \mathbb{H}_T \vee \bar{\mathbb{L}}_{T_n} \right] = 0, \quad \forall n \in I(0, \infty) \\
& Z_{T_n} x_n E \left[\frac{\beta_n e^{\alpha_n (\beta_n - 1)}}{E[e^{\alpha_n (\beta_n - 1)} \mid \theta_{n-1}, \theta_n]} \mid \theta_{n-1}, \theta_n \mid \mathbb{H}_T \vee \bar{\mathbb{L}}_{T_n} \right] - 1 = 0, \quad \forall n \in I(0, \infty)
\end{aligned}$$

$$\begin{aligned}
 & E \left[\frac{\beta_n e^{\alpha_n(\beta_n-1)}}{E[e^{\alpha_n(\beta_n-1)} | \theta_{n-1}, \theta_n]} \middle| \theta_{n-1}, \theta_n \right] - 1 = 0, \quad \forall n \in I(0, \infty), \\
 & E \left[\frac{\beta_n \exp[\int_{T_n}^{T_{n+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz)]}{E[\exp[\int_{T_n}^{T_{n+1}} \int_{z \in \mathbb{R}} k(s, z, ds, dz) | \theta_{n-1}, \theta_n]]} \middle| \theta_{n-1}, \theta_n \right] - 1 = 0, \quad \forall n \in I(0, \infty), \\
 & EP^\alpha[\beta_n | \theta_{n-1}, \theta_n] - 1 = 0, \quad \forall n \in I(0, \infty).
 \end{aligned} \tag{6.35}$$

This completes the proof of (1). (2) is a direct consequence of (1) whenever $\mu(\theta_{s-})$ is replaced by $\mu(\theta_{s-}) - r_s$. For the proof of (3), we use (6.27) to derive the risk neutral dynamic of the process R^θ defined in (6.16). We denote $B^{P^{\alpha^*}}$ and $\nu^{P^{\alpha^*}}$ the standard Brownian motion and the intensity process of the Poisson process ψ under the probability measure P^{α^*} , respectively. From Lemma 6.2 (5(a)), solving for B in $B_t^{P^{\alpha^*}} = B_t - \int_0^t \alpha_{s-}^* \sigma(\theta_{s-}) ds$, adding and subtracting $\nu^{P^{\alpha^*}}$ inside the Poisson integrals, we obtain:

$$\begin{aligned}
 dR_t^\theta &= \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} (e^z - 1) N(ds, dz, \{(\theta_{t-}, j)\}) + \mu(\theta_{t-}) dt + \sigma(\theta_{t-}) dB_t \\
 &\quad + \int_{|z| \leq 1} G(z, \theta_{t-}) \bar{\psi}(\theta_{t-}, dz, dt) + \int_{|z| > 1} H(z, \theta_{t-}) \psi(\theta_{t-}, dz, dt) \\
 &= \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} (e^z - 1) N(dt, dz, \{(\theta_{t-}, j)\}) + \left[\mu(\theta_{t-}) + \sigma^2(\theta_{t-}) \alpha_{t-}^* \right. \\
 &\quad + \int_{|z| \leq 1} G(z, \theta_{t-}) [\nu^{P^{\alpha^*}}(\theta_{t-}, dz) - \nu(\theta_{t-}, dz)] \\
 &\quad \left. + \int_{|z| > 1} H(z, \theta_{t-}) \nu^{P^{\alpha^*}}(\theta_{t-}, dz) \right] dt + \sigma(\theta_{t-}) dB^{P^{\alpha^*}} \\
 &\quad + \int_{|z| \leq 1} G(z, \theta_{t-}) [\psi(\theta_{t-}, dt, dz) - \nu^{P^{\alpha^*}}(\theta_{t-}, dz) dt] \\
 &\quad + \int_{|z| > 1} H(z, \theta_t) [\psi(\theta_t, dt, dz) - \nu^{P^{\alpha^*}}(\theta_t, dz) dt], \quad \forall t \in [T_n, T_{n+1}]. \tag{6.36}
 \end{aligned}$$

From Lemma 6.2 (5(a)), one gets $\nu^{P^\alpha}(j, dz) = e^{[H(z, j)1_{|z| > 1} + G(z, j)1_{|z| \leq 1}]} \nu(j, dz)$. Hence,

$$\begin{aligned}
 dR_t^\theta &= \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} (e^z - 1) N(dt, dz, \{(\theta_{t-}, j)\}) + \left[\mu(\theta_{t-}) dt + \sigma^2(\theta_{t-}) \alpha_{t-}^* \right. \\
 &\quad \left. + \int_{|z| \leq 1} G(z, \theta_{t-}) [e^{\alpha_{t-}^* G(z, \theta_{t-})} - 1] \nu(\theta_{t-}, dz) dt + \sigma(\theta_{t-}) dB^{P^{\alpha^*}} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{|z|>1} H(z, \theta_{t-}) e^{\alpha_{\theta_{t-}}^* H(z, \theta_{t-})} \nu(\theta_{t-}, dz) dt \Big] \\
 & + \int_{|z|\leq 1} G(z, \theta_{t-}) [\psi(\theta_{t-}, dt, dz) - \nu^{\alpha^*}(\theta_{t-}, dz) dt] \\
 & + \int_{|z|>1} H(z, \theta_t) [\psi(\theta_t, dt, dz) - \nu^{P^{\alpha^*}}(\theta_t, dz) dt], \tag{6.37}
 \end{aligned}$$

where α^* satisfies the condition (1). Therefore, (6.37) becomes

$$\begin{aligned}
 dR_t^\alpha &= \int_{z \in \mathbb{R}} \sum_{j \in E \setminus \{\theta_{t-}\}} (e^z - 1) N(ds, dz, \{(\theta_{t-}, j)\}) + \sigma(\theta_{t-}) dB^{P^{\alpha^*}} \\
 &+ \int_{|z|\leq 1} G(z, \theta_{t-}) [\psi(\theta_{t-}, dt, dz) - \nu^{P^{\alpha^*}}(\theta_{t-}, dz) dt] \\
 &+ \int_{|z|>1} H(z, \theta_{t-}) [\psi(\theta_{t-}, dt, dz) - \nu^{P^{\alpha^*}}(\theta_{t-}, dz) dt], \quad \forall t \in [T_n, T_{n+1}], \tag{6.38}
 \end{aligned}$$

with

$$E_{P^\alpha}[\beta_n | \theta_{n-1}, \theta_n] = 1,$$

which proves (3). This establishes the lemma. \square

In the next remark, we introduce a particular case of R_t^θ corresponding to the simple return process Miyahara *et al.* (2001) and we present a few properties of conditional entropies Fujiwara (2003).

Remark 6.3. Let P_1 and P_2 be two absolutely continuous probability measures relative to P . We recall three important properties of conditional entropies Miyahara (1999)

- (1) $\mathcal{H}_{\mathbb{H}_T \vee \mathbb{L}_t}^{\mathbb{H}_T}(P_1 | P) \geq 0$.
- (2) $\mathcal{H}_{\mathbb{G}}^{\mathbb{H}_T}(P_1 | P) \leq \mathcal{H}_{\mathbb{K}}^{\mathbb{H}_T}(P_1 | P)$, if $K \subset \mathbb{H}_T \vee \mathbb{L}_T$.
- (3) If P_1 is a $(P, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -absolutely continuous martingale measure, and P_2 is a probability measure equivalent to P such that $\ln(\frac{dP_2}{dP})$ is integrable with respect to P_1 , then $\mathcal{H}_{\mathbb{H}_T \vee \mathbb{L}_T}^{\mathbb{H}_T}(P_1 | P) \geq E_{P_1}[\ln(dP_2 | dP) | \mathbb{H}_T]$.

We now state and prove the conditional minimum entropy property of the martingale measure P^{α^*} when R_t^θ is the simple return process of x_t in Remark 6.3.

Lemma 6.4. Let $Q \in \mathcal{M}(\tilde{x}, P) = \{Q \ll P : \tilde{x} \text{ is a } (Q, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in (0, T)})\text{-Local martingale}\}$. Let P^{α^*} be defined as in Definition 6.1 with α^* solution process of (6.28). Then the following inequality holds:

$$\mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}^{\mathbb{H}_T}(Q | P) \geq \mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}^{\mathbb{H}_T}(P^{\alpha^*} | P), \quad \forall Q \in \mathcal{M}(\tilde{x}, P),$$

where $\mathcal{H}_{\mathbb{L}_T \vee \mathbb{H}_T}^{\mathbb{H}_T}$ is the conditional entropy defined in Elliott *et al.* (2005).

Proof. We prove the lemma in two steps. The first step consists in minimizing the conditional relative entropy of any probability measure Q in the set $\mathcal{M}(\tilde{x}, P)$. From (6.18) and Remark 6.3, one notes that Z_t^α can also be expressed as follows:

$$Z_t^\alpha = \frac{e^{\int_0^t \alpha_s - d\tilde{R}_s^\theta}}{E[e^{\int_0^t \alpha_s - d\tilde{R}_s^\theta} | \mathbb{H}_T]}, \quad \forall t \geq 0. \quad (6.39)$$

By definition of an absolutely continuous local martingale measure, the discounted stock price $\tilde{x} = e^{-\int_0^t r_s - ds} x_t$ is a $(Q, \mathbb{H}_T \vee \tilde{\mathbb{L}}_t)_{t \in [0, T]}$ -local martingale process. The simple return processes R_t and \tilde{R}_t are associated with the price process x_t and the discounted price process \tilde{x}_t with respect to (4.1), defined by $dR_t^\theta = \frac{dx_t}{x_t}$ and $d\tilde{R}_t^\theta = \frac{d\tilde{x}_t}{\tilde{x}_t}$, respectively. $Q \in \mathcal{M}(\tilde{x}, P)$ implies that \tilde{x} is a $(P, \mathbb{H}_T \vee \tilde{\mathbb{L}}_t)$ -martingale. Hence, \tilde{R}_t^θ is a Q -local martingale process. Furthermore, $\int_0^t \alpha_s - d\tilde{R}_s^\theta$ is a local Q -martingale. As a Q -local martingale, $\int_0^t \alpha_s^* - d\tilde{R}_s^\theta$ is therefore integrable with respect to Q . From (6.39) and (6.21), we have

$$\ln(Z_t^{\alpha^*}) = \int_0^t \alpha_s^* - d\tilde{R}_s^\theta - \int_0^t g(s) ds, \quad (6.40)$$

where

$$g(t) = \ln(E[e^{\int_0^t \alpha_s^* - d\tilde{R}_s^\theta} | \mathbb{H}_T]). \quad (6.41)$$

From (6.40), $\ln(Z_t^{\alpha^*})$ is integrable with respect to Q as a sum of two integrable terms. Let $(t_n)_{n \in I(0, \infty)}$ be a local sequence of increasing stopping times with $\lim_{n \rightarrow \infty} t_n = T$, associated with the local martingale $\int_0^t \alpha_s - d\tilde{R}_s^\theta$. By definition of local sequences, the process $\int_0^{t_n \wedge t} \alpha_s - d\tilde{R}_s^\theta$ is a Q -martingale. Hence, for any $t \in [0, T]$, we have

$$\begin{aligned} \mathcal{H}_{\mathbb{H}_T \vee \tilde{\mathbb{L}}_T}^{\mathbb{H}_T}(Q | P) &\geq \mathcal{H}_{\mathbb{H}_T \vee \tilde{\mathbb{L}}_{t \wedge t_n}}^{\mathbb{H}_T}(Q | P), \quad (\text{Remark 6.3}) \\ &\geq E_Q \left[\ln \left(\frac{dP^{\alpha^*}}{dP} \Big|_{\mathbb{H}_T \vee \tilde{\mathbb{L}}_{t \wedge t_n}} \right) \Big| \mathbb{H}_T \right], \quad (\text{Remark 6.3}). \end{aligned} \quad (6.42)$$

From (6.40), we have

$$\begin{aligned} &E_Q \left[\ln \left(\frac{dP^{\alpha^*}}{dP} \Big|_{\mathbb{H}_T \vee \tilde{\mathbb{L}}_{t \wedge t_n}} \right) \Big| \mathbb{H}_T \right] \\ &= E_Q[\ln(Z_{t \wedge t_n}^{\alpha^*}) | \mathbb{H}_T] \\ &= E_Q \left[\int_0^{t \wedge t_n} \alpha_s - d\tilde{R}_s^\theta \Big| \mathbb{H}_T \right] + E_Q[g(t \wedge t_n) | \mathbb{H}_T] \\ &= E_Q \left(\int_0^{t \wedge t_n} \alpha_s d\tilde{R}_s^\theta \Big| \mathbb{H}_T \vee \tilde{\mathbb{L}}_0 \right) + E_Q[g(t \wedge t_n) | \mathbb{H}_T] \\ &= E_Q[g(t \wedge t_n) | \mathbb{H}_T], \end{aligned}$$

since $\int_0^{t \wedge t_n} \alpha_s - d\tilde{R}(s)$ is a $(\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t \in [0, T]}$ -martingale. We note that, $|E_Q[g(t \wedge t_n) | \mathbb{H}_T]| \leq E_Q[|g(T)| | \mathbb{H}_T] = g(T), \forall t \in [0, T]$, since from (6.41), g is \mathbb{H}_T -measurable. Hence, by the Dominated Convergence theorem, we have

$$\lim_{n \rightarrow \infty} E_Q[g(t \wedge t_n) | \mathbb{H}_T] = E_Q[g(T) | \mathbb{H}_T] = |g(T)|. \quad (6.43)$$

Taking the limit in (6.42), we obtain

$$\mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}^{\mathbb{H}_T}(Q | P) \geq E_Q[g(T) | \mathbb{H}_T] = g(T). \quad (6.44)$$

The second step of the proof consists in showing that the conditional relative entropy of the random Esscher transform achieves the minimum value in (6.44). Using (6.39) and the P^{α^*} -martingale property of \tilde{R}_t the relative entropy of P^{α^*} is computed as follows:

$$\begin{aligned} \mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}^{\mathbb{H}_T}(P^{\alpha^*} | P) &= E_{P^{\alpha^*}} \left[\ln \left(\frac{dP^{\alpha^*}}{dP} \right) \middle| \mathbb{H}_T \right] \\ &= E_{P^{\alpha^*}} \left[\int_0^T \alpha_s - d\tilde{R}_s \middle| \mathbb{H}_T \vee \bar{\mathbb{L}}_0 \right] \\ &\quad + E_{P^{\alpha^*}}[g(T) | \mathbb{H}_T], \quad (\text{from (6.39)}) \\ &= E_{P^{\alpha^*}}[g(T) | \mathbb{H}_T] = g(T), \quad (\tilde{R}_t^\theta \text{ is a } P^{\alpha^*} \text{ martingale}). \end{aligned}$$

From (6.44), the lemma follows. \square

6.2. Unconditional minimum entropy martingale measure

We will define an equivalent martingale probability measure and we will establish that it has the unconditional minimum entropy martingale measure property. $(\lambda_{i,j}(t))_{m \times m}$ is the intensity matrix of the semi-Markov process θ from (2.9) and N is the point process defined in Definition 4.4.

Definition 6.2. Let Q be a local absolutely continuous probability measure with respect to the historical probability measure P on the filtered measurable space $(\Omega, \mathbb{H}_T \vee \bar{\mathbb{L}}_T, (\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$; \hat{Q} and \hat{P} , denote the regular versions of the conditional probabilities $P(|\mathbb{H}_T)$ and $Q(|\mathbb{H}_T)$ over $\mathbb{H}_T \vee \bar{\mathbb{L}}_T$. Z_t^Q denotes a local martingale process with initial value 1, representing the density process of Q with respect to P . $(\xi_{i,j}(t))_{m \times m}$ denotes a matrix entries with predictable processes. Moreover, the rows add up to 0 and satisfy $\sum_{(i,j) \in E^2} \int_0^t |\xi_{i,j}(s) \lambda_{i,j}(s)| ds < \infty$. N and γ are the processes defined in Definition 4.4.

We first recall a decomposition theorem Momeya (2012) and we establish a Girsanov-type lemma necessary in the proof of the UMEMM property.

Lemma 6.5. Let $Q, P, Z_t^Q, \hat{Q}(w, \cdot), (\xi_{i,j})_{m \times m}, N, \bar{N}$ and $\hat{P}(w, \cdot)$ be processes and probability measures defined in Definitions 6.2 and 4.4. The following

claims hold:

- (1) There exist two density processes Z_t^L and Z_T^H such that

$$Z_t^Q = Z_t^L \times Z_T^H \quad (6.45)$$

with

$$\left. \frac{d\hat{Q}}{d\hat{P}} \right|_{\bar{\mathbb{I}}_t \vee \mathbb{H}_T} = Z_t^L \quad \text{and} \quad \left. \frac{dQ}{dP} \right|_{\mathbb{H}_T} = Z_T^H.$$

- (2) If $(\lambda_{i,j}(t))_{m \times m}$ is the matrix with conditional intensity of the semi-Markov process θ in (2.9) and $\lambda_{i,j}(t) \neq 0, \forall t \in [0, T]$, then the following claims are equivalent:

(a)

$$\left. \frac{dQ}{dP} \right|_{\mathbb{H}_t} = Z_t^H, \quad \text{where } Z^H \text{ solves the SDE:}$$

$$dZ_t^H = Z_{t-}^H \sum_{(i,j) \in E^2} \left[-1 + \frac{\xi_{i,j}(t^-)}{\lambda_{i,j}(t^-)} \right] \bar{N}(dt, \mathbb{R}, \{(i,j)\}), \quad Z_0^H = 1.$$

- (b) Under probability measure Q , with density process Z_t^H , the point process M has conditional intensities matrix $(\xi_{i,j}(t))_{m \times m}$.

Proof. The proof of (1) follows closely Momeya (2012). As for (2), we note that (2(a)) \Rightarrow (2(b)) follows from the proof of Lemma 6.1. We now aim at proving that (2(b)) \Rightarrow (2(a)). From Definition 6.2 and (2(b)), Z_t^H and $N(\mathbb{R}, \{(i,j)\}) - \gamma(\mathbb{R}, \{(i,j)\})$ are $(P, (\mathbb{H}_t)_{t \in [0, T]})$ -martingale processes. From the martingale representation property of $\bar{N}(t, \mathbb{R}, \{(i,j)\}) = N(t, \mathbb{R}, \{(i,j)\}) - \gamma(\mathbb{R}, \{(i,j)\}) = N(t, \mathbb{R}, \{(i,j)\}) - \lambda_{i,j}(t)$, there exists an $m \times m$ matrix of \mathbb{H}_t -predictable processes $(s_t^{i,j})_{m \times m}$ such that

$$dZ_t^H = \sum_{(i,j) \in E^2} s_t^{i,j} \bar{N}(dt, \mathbb{R}, \{(i,j)\}).$$

As $Z_t^H > 0$ P -almost surely, there exists an $m \times m$ matrix of predictable processes $\tilde{s}_t^{i,j}$ satisfying $s_t^{i,j} = Z_t^H \tilde{s}_t^{i,j}$. Hence,

$$dZ_t^H = Z_{t-}^H \sum_{(i,j) \in E^2} \tilde{s}_t^{i,j} \bar{N}(dt, \mathbb{R}, \{(i,j)\}).$$

From Lemma 6.1, the matrix of conditional Q -intensities of $N(\mathbb{R}, \{(i,j)\})$ is $\lambda_{i,j}(t)(1 + \tilde{s}_t^{i,j})$. One needs to prove that the conditional intensity of $N(\mathbb{R}, \{(i,j)\})$ with respect to Q is $\xi_{i,j}(t), \forall i, j \in I(1, m)$. Hence, equating both matrices and solving for $\tilde{s}_t^{i,j}$ yields

$$\lambda_{i,j}(t)(1 + \tilde{s}_t^{i,j}) = \xi_{i,j}(t), \quad \forall t \in [0, T], \quad (i, j) \in E^2, \quad \text{and hence,}$$

$$\tilde{s}_t^{i,j} = \left[-1 + \frac{\xi_{i,j}(t)}{\lambda_{i,j}(t)} \right], \quad \forall t \in [0, T], \quad (i, j) \in E^2.$$

Therefore, Z^H is solution process of the SDE

$$dZ_t^H = Z_t^H \sum_{(i,j) \in E^2} \left[-1 + \frac{\xi_{i,j}(t^-)}{\lambda_{i,j}(t^-)} \right] [N(dt, \mathbb{R}, \{(i,j)\}) - \xi_{i,j}(t^-)dt] \quad Z_0 = 1.$$

From Lemma 6.1, the intensity matrix of the semi-Markov process θ under the probability measure Q is $(\xi_{i,j}(t))_{m \times m}$. This completes the proof of (2) and hence the lemma. \square

We define a density process which we prove is the unconditional minimum entropy martingale measure.

Definition 6.3. Let $P^{\alpha^*, \xi}$ be a risk neutral measure with density $Z_t = Z_t^{\alpha^*} \times Z_t^\xi$, where Z^{α^*} is introduced in Definition 6.1, with R^θ the simple return process of defined in (6.16). α^* is the solution process of (6.28) and Z_t^ξ is solution of the SDE $dZ_t^\xi = Z_t^\xi \sum_{(i,j) \in E^2} \tilde{s}_t^{i,j} d\bar{M}_t^{i,j}$.

$$\begin{aligned} \frac{dP^{(\alpha^*, \xi)}}{dP} \Big|_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T} &= \frac{e^{\int_0^T \alpha_{s^-}^* dR_s}}{E(e^{\int_0^T \alpha_{s^-}^* dR_s} | \mathbb{H}_T)} \prod_{(i,j) \in E^2} \exp \left[\int_0^T (1 - \xi_{i,j}(s^-)) \lambda_{i,j}(s^-) ds \right. \\ &\quad \left. + \int_0^T \ln(\xi_{i,j}(s^-)) N(ds, \mathbb{R}, \{(i,j)\}) \right]. \end{aligned}$$

We also define a functional F as follows:

$$\begin{aligned} F((\xi_{i,j})) &= E \left[g(T) + \sum_{(i,j) \in E^2} \left[\int_0^T (1 - \xi_{i,j}(s^-)) \lambda_{i,j}(s^-) ds \right. \right. \\ &\quad \left. \left. + \int_0^T \ln(\xi_{i,j}(s^-)) N(ds, \mathbb{R}, \{(i,j)\}) \right] \right. \\ &\quad \left. \prod_{(i,j) \in E^2} \exp \left[\int_0^T (1 - \xi_{i,j}(s^-)) \lambda_{i,j}(s^-) ds \right. \right. \\ &\quad \left. \left. + \int_0^T \ln(\xi_{i,j}(s^-)) N(ds, \mathbb{R}, \{(i,j)\}) \right] \right], \end{aligned}$$

where g is defined in (6.41).

We will next show that under a particular choice of ξ , $P^{\alpha^*, \xi}$ has the unconditional minimum entropy martingale measure property.

Lemma 6.6. We denote $P^{\alpha^*, \bar{\xi}}$ and F the risk neutral measure and the functional from Definition 6.3, respectively. If Q is a $(\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t \in [0, T]}$ risk neutral measure and

$(\bar{\xi}_t^{i,j})_{m \times m}$ minimizes the functional F , then the following holds:

$$\mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(Q | P) \geq \mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(P^{(\alpha^*, \bar{\xi})} | P).$$

Proof. Let Q be a risk neutral measure. By definition of risk neutral measures, Q is locally absolutely continuous with respect to P . From Lemma 6.5, there exists a process Z_t^L and a process Z_t^H such that $\frac{dQ}{dP}|_{\mathbb{H}_T \vee \bar{\mathbb{L}}_t} = Z_t^L \times Z_t^H$. From Lemma 6.5, we have

$$Z_t^H = \prod_{(i,j) \in E^2} \exp \left[\int_0^t (1 - \xi_{i,j}(s^-)) \lambda_{i,j}(y_{s^-}) ds + \int_0^t \ln(\xi(s^-)) N(ds, \mathbb{R}, \{(i,j)\}) \right],$$

for some $m \times m$ matrix of \mathbb{H}_t -predictable processes $\xi_{i,j}(s)$ as in Definition 6.2.

$$\begin{aligned} \mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(Q | P) &= E \left[\ln \left(\frac{dQ}{dP} \right) \frac{dQ}{dP} \right] \\ &= E[Z_T^L Z_T^H (\ln(Z_T^L)) + Z_T^L Z_T^H \ln(Z_T^H)] \\ &= E[E[Z_T^L Z_T^H \ln(Z_T^L) | \mathbb{H}_T] + E[Z_T^L Z_T^H \ln(Z_T^H) | \mathbb{H}_T]] \\ &= E[Z_T^H E[Z_T^L \ln(Z_T^L) | \mathbb{H}_T] + Z_T^H \ln(Z_T^H)] \\ &= E[Z_T^H \mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}^{\mathbb{H}_T}(\hat{Q} | \hat{P}) + Z_T^H \ln(Z_T^H)] \\ &\geq E[Z_T^H \mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}^{\mathbb{H}_T}(\bar{P}^\alpha | \hat{P}) + Z_T^H \ln(Z_T^H)] \\ &= E \left[g(T) + \sum_{(i,j) \in E^2} \left[\int_0^T (1 - \xi_{i,j}(s^-)) \lambda_{i,j}(y_{s^-}) ds \right. \right. \\ &\quad \left. \left. + \int_0^T \ln(\xi_{i,j}(s^-)) N(ds, \mathbb{R}, \{(i,j)\}) \right] \right. \\ &\quad \times \prod_{(i,j) \in E^2} \exp \left[\int_0^T (1 - \xi_{i,j}(s^-)) \lambda_{i,j}(s^-) ds \right. \\ &\quad \left. \left. + \int_0^T \ln(\xi_{i,j}(s^-)) N(ds, \mathbb{R}, \{(i,j)\}) \right] \right] \\ &= F(\xi_{i,j}) \\ &\geq F(\bar{\xi}_{i,j}), \quad (\text{definition of } \bar{\xi}) \\ &= E \left[\ln \left(\frac{d\bar{P}^{\alpha^*, \bar{\xi}}}{dP} \right) \frac{d\bar{P}^{\alpha^*, \bar{\xi}}}{dP} \right] \\ &= \mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(\bar{P}^{\alpha^*, \bar{\xi}} | P), \end{aligned}$$

which proves the result. \square

6.3. Siu and Yang Kernel pricing all risks

Let α be a piecewise constant stochastic process. We define the density process Z_t^α of a probability measure, P^α , on the filtration $(\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t \in [0, T]}$, with an Esscher transform with parameter α . The pricing kernel discussed here is based on the work in Siu & Yang (2009), in the context of a Markov switching asset price process.

Definition 6.4. Let Z^α be the following stochastic process:

$$\frac{d\bar{P}^\alpha}{dP} \Big|_{\mathbb{H}_t \vee \bar{\mathbb{L}}_t} = Z_t^\alpha = \begin{cases} E \left[\frac{e^{-\int_0^T \alpha(s) dR_s^\theta}}{E[e^{-\int_0^T \alpha_s - dR_s^\theta} | \theta_0, y_0]} \Big| \mathbb{H}_t \vee \bar{\mathbb{L}}_t \right] & \text{if } \forall t \in (0, T] \\ 1 & \text{if } t = 0, \end{cases}$$

where R_t^θ is the log price process in (6.16), induced by x from (4.1) in the context of solution (4.2).

We note that from Lemma 5.1, one can retrieve any particular scalar conditional characteristic function from the vector characteristic function as follows: $\Psi(u, t, y, j, x) = \exp[iu \ln(x)] \langle \exp(\int_y^{t+y} M(u, s) ds) \cdot e_j, \mathbf{1} \rangle$, where $e_{\theta_t} = (1_{\theta_t=1}, 1_{\theta_t=2}, \dots, 1_{\theta_t=m})^\top$. α_t is the Esscher parameter process associated with the probability measure \bar{P}^α .

Lemma 6.7. Let Z_t^α be the process in Definition 6.4. Z_t^α is an almost surely positive martingale with unitary expectation.

Proof. We first prove that Z_t^α is a martingale. Let $0 \leq s \leq t$

$$\begin{aligned} E[Z_t^\alpha | \mathbb{H}_s \vee \bar{\mathbb{L}}_s] &= E \left[E \left[\frac{e^{-\int_0^T \alpha_s - dR_s^\theta}}{E[e^{-\int_0^T \alpha_s - dR_s^\theta} | y_0, \theta_0, L_0]} \Big| \mathbb{H}_t \vee \bar{\mathbb{L}}_t \right] \Big| \mathbb{H}_s \vee \bar{\mathbb{L}}_s \right] \\ &= E \left[\frac{e^{-\int_0^T \alpha_s - dR_s^\theta}}{E[e^{-\int_0^T \alpha_s - dR_s^\theta} | y_0, \theta_0, L_0]} \Big| \mathbb{H}_s \vee \bar{\mathbb{L}}_s \right], \quad (\mathbb{H}_s \vee \bar{\mathbb{L}}_s \subset \mathbb{H}_t \vee \bar{\mathbb{L}}_t) \\ &= Z_s^\alpha. \end{aligned}$$

Therefore, Z_t^α is a martingale. It follows that Z_t^α has unitary expectation,

$$E(Z_t^\alpha) = E[E(Z_t^\alpha) | \mathbb{H}_0 \vee \bar{\mathbb{L}}_0] = E(Z_0^\alpha) = 1.$$

Noting that Z_t^α is an almost surely positive process by construction, the lemma follows. \square

From the preceding lemma, Z_t^α is a density process. Hence, the Esscher transform in (6.4) defines a probability measure \bar{P}^α equivalent to P . It remains to show that \bar{P}^α is a martingale measure under a certain condition specified in the next lemma.

Lemma 6.8. Let Z_t^α be from Definition 6.4 and x as defined in (4.1). $M(u, y)$ and $\bar{M}(u, y)$ are defined in (5.4) with modified log price process defined by

$dR_t^\theta = \alpha_{t-} \ln(\beta_{n(t)}) + \alpha_{t-} dL_t^\theta$ and $dR_t^\theta = \alpha_{t-} \ln(\beta_{n(t)}) - r_{t-} dt + (\alpha_{t-} + 1) dL_t^\theta$, respectively, with L^θ defined in (3.2).

$\tilde{x}_t = e^{-\int_0^t r_s - ds} x_t$ is a $(\bar{P}^\alpha, (\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale if and only if

$$\begin{aligned} & \left\langle \exp \left[\int_y^{t+y} \bar{M}(-i, s) ds \right] \cdot e_{\theta_0}, \mathbf{1} \right\rangle \\ & - \left\langle \exp \left[\int_y^{t+y} M(-i, s) ds \right]^\top \cdot e_{\theta_0}, \mathbf{1} \right\rangle = 0 \quad \forall t \in [0, T], \quad \theta_u \in E. \end{aligned} \quad (6.46)$$

Proof. Let $0 \leq u \leq t$. From Momeya (2012), Siu & Yang (2009) and by the abstract Bayes rule (Jacod & Shiryaev 1987), we have

$$E(Z_t^\alpha e^{-\int_0^t r_s - ds} x_t \mid \mathbb{H}_u \vee \bar{\mathbb{L}}_u) \quad (6.47)$$

$$= e^{-\int_0^u r_s - ds} x_u \frac{E[e^{-\int_u^t r_s - ds + \int_u^t (\alpha(s^-) + 1) dR_s^\theta} \mid \mathbb{H}_u \vee \bar{\mathbb{L}}_u]}{E[e^{\int_u^t \alpha_s - dR_s^\alpha} \mid \mathbb{H}_u \vee \bar{\mathbb{L}}_u]}. \quad (6.48)$$

Hence, $e^{-\int_0^t r_s - ds} x_t$ is a $(\bar{P}^\alpha, (\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t \in [0, T]})$ -martingale if and only if

$$\frac{E[e^{-\int_u^t r_s - ds + \int_0^t (\alpha_s - + 1) dR_s^\alpha} \mid \mathbb{H}_u \vee \bar{\mathbb{L}}_u]}{E[e^{\int_0^t \alpha_s - dR_s^\alpha} \mid \mathbb{H}_u \vee \bar{\mathbb{L}}_u]} = 1 \quad \forall u, t \in [0, T]. \quad (6.49)$$

From Lemma 5.1 applied to $dR_t^\theta = \alpha_{t-} \ln(\beta_{n(t)}) + \alpha_{t-} dL_t^\theta$ and $dR_t^\theta = \alpha_{t-} \ln(\beta_{n(t)}) - r_{t-} dt + (\alpha_{t-} + 1) dL_t^\theta$, respectively and on account of the Markov property and the homogeneity of the process (θ, y) , the numerator and the denominator of (6.49) becomes

$$\begin{aligned} & E[e^{-\int_u^t r_s - ds + \int_0^t (\alpha(s^-) + 1) dL_s^\alpha} \mid \mathbb{H}_u \vee \bar{\mathbb{L}}_u] \\ & = \left\langle \exp \left(\int_y^{y+t-u} \bar{M}(-i, s) ds \right) \cdot e_{\theta_0}, \mathbf{1} \right\rangle \end{aligned}$$

and

$$E[e^{\int_0^t \alpha(s^-) dL_s^\alpha} \mid \mathbb{H}_u \vee \bar{\mathbb{L}}_u] = \left\langle \exp \left(\int_y^{y+t-u} M(-i, s) ds \right) \cdot e_{\theta_0}, \mathbf{1} \right\rangle,$$

respectively, where,

$$M_{p,q}(-i, y)$$

$$= \begin{cases} -r(q) + (\alpha_q + 1)\mu(q) + \frac{1}{2}(\alpha_q + 1)^2 \sigma^2(q) + \int_{|z| > 1} [e^{(\alpha_q + 1)G(z, q)} - 1] \nu(q, dz) \\ \quad + \int_{|z| \leq 1} [e^{(\alpha_q + 1)G(z, q)} - 1 - (\alpha_q + 1)G(z, q)] \nu(q, dz) + \lambda_{q,q}(y) & \text{if } p = q \\ \lambda_{q,p}(y) \int_{z \in \mathbb{R}} e^{(\alpha_q + 1)z} \bar{b}(z \mid q, p) dz & \text{otherwise} \end{cases}$$

and

$$M_{p,q}(-i, y) = \begin{cases} \alpha_q \mu(q) + \frac{1}{2} \alpha_q^2 \sigma^2(q) + \int_{|z| \leq 1} [e^{\alpha_q G(z, q)} - 1 - \alpha_q G(z, q)] \nu(q, dz) \\ \quad + \int_{|z| > 1} [e^{\alpha_q G(z, q)} - 1] \nu(q, dz) + \lambda_{q,q}(y) & \text{if } p = q \\ \lambda_{q,p}(y) \int_{z \in \mathbb{R}} e^{\alpha_q z} \bar{b}(z | q, p) dz & \text{otherwise.} \end{cases}$$

Hence, (6.49) becomes

$$\begin{aligned} & \left\langle \exp \left[\int_y^{y+t-u} \bar{M}(-i, s) ds \right] \cdot e_{\theta_0}, \mathbf{1} \right\rangle \\ & - \left\langle \exp \left[\int_y^{y+t-u} M(-i, s) ds \right] \cdot e_{\theta_0}, \mathbf{1} \right\rangle = 0, \quad \forall u, t \in [0, T], \quad \forall \theta_u \in E, \end{aligned} \quad (6.50)$$

which completes the proof of the lemma. \square

7. Option Pricing Formulas

In this section, we price a European style call option within the risk neutral pricing theory Schachermayer (2010). We denote Q an equivalent martingale measure of the historical probability measure P , relative to the price process x in (4.1). We derive a PIDE extending the PDE in Black & Scholes (1973) satisfied by European call prices. We also describe how two existing pricing methods blend seamlessly in the context of this paper.

Definition 7.1. Let S be a function in $L^2(\Omega, Q)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ into \mathbb{R} representing the payoff of a contingent claim; Q is a risk neutral probability measure of the price process x defined by (4.1) with respect to the historical probability measure P ; K is a non-negative real number denoting the strike price of a European type option contract with maturity T ; x_T denotes the asset price value at maturity; C is the Q -risk neutral option price function defined on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T] \times E \times \mathbb{R}^+$ into \mathbb{R}^+ and V denotes the discounted option price process defined by $V(t, T, K, y_t, \theta_t, x_t) = e^{-\int_0^t r_s ds} C(t, T, K, y_t, \theta_t, x_t)$.

Lemma 7.1. *Let S be a random variable representing the payoff of a general European style contingent claim with maturity T and strike price K in Definition 7.1; let Q be the risk neutral measure defined in Definition 7.1 and C is the Q -risk neutral option price of a contingent claim. Then, the Q -risk neutral option price C of a European contingent claim with maturity T , strike price K and payoff S can be expressed*

$$C(t, T, K, y_t, \theta_t, x_t) = E_Q(e^{-\int_t^T r_s ds} S(x_T, K) | y_t, \theta_t, x_t). \quad (7.1)$$

Proof. From Schachermayer (2010), the Q -risk neutral option price C at time t is given by

$$C(t, T, K, y_t, \theta_t, x_t) = E_Q[e^{-\int_t^T r_s ds} S(x_T, K) \mid \mathbb{H}_t \vee \bar{\mathbb{L}}_t].$$

We note from Lemma 2.3 that the triplet (y, θ, x) is Markovian, hence

$$C(t, T, K, y_t, \theta_t, x_t) = E_Q[e^{-\int_t^T r_s ds} S(x_T, K) \mid y_t, \theta_t, x_t],$$

which proves the result. \square

A partial integro differential equation (PIDE) satisfied by a European style contingent claim with maturity T and payoff H is presented in the next lemma.

Lemma 7.2. *Let Q , C and V be the risk neutral measure, the Q -risk neutral option price function and the discounted option price process defined in 7.1, respectively. Then V satisfies the following system of PIDE*

$$\begin{aligned} & \frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(j)x_{s-} \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(j)x_s^2 \frac{\partial^2 V}{\partial x^2} \\ & + \int_{|z| \leq 1} \left[V(s, T, K, y_s, \theta_s, x_{s-} + x_{s-}G(z, j)) \right. \\ & \quad \left. - V(s, T, K, y_s, j, x_{s-}) - G(z, j)x_{s-} \frac{\partial V}{\partial x} \right] \nu(j, dz) \\ & + \int_{|z| > 1} [V(s, T, K, y_s, \theta_s, x_{s-} + x_{s-}H(z, \theta_s)) \\ & \quad - V(s, T, K, y_s, \theta_s, x_{s-})] \nu(\theta_s, dz) \\ & + \int_{z \in \mathbb{R}} \sum_{j \neq i} \lambda_{i,j}(y_s) V(s, T, K, y_s, j, x_{s-}e^z) \bar{b}(z \mid i, j) dz \\ & + V(s, T, K, y_{s-}, i, x_{s-}) \lambda_{j,j}(y) = 0, \end{aligned}$$

with terminal condition

$$V(T, T, K, y_T, \theta_T = j, x_T) = e^{-\int_0^T r(\theta_s) ds} S(x_T, K), \quad \text{for } j \in E.$$

Proof. From (7.1), the discounted price process could be expressed as follows:

$$\begin{aligned} V(t, T, K, y, j, x) &= e^{-\int_0^t r(\theta_s) ds} C(t, T, K, y, j, x) \\ &= E^Q(e^{-\int_{[0,T]} r(\theta_s) ds} S(x_T, K) \mid y_t, \theta_t, x_t) \end{aligned} \quad (7.2)$$

V is a $(Q, \bar{\mathbb{L}}_t \vee \mathbb{H}_t)$ -Martingale since it is a Q -conditional expectation. We use the law of iterated expectation and $u \leq t$ to prove it as follows:

$$\begin{aligned} & E(V(t, T, K, y, j, x) \mid \bar{\mathbb{L}}_u \vee \mathbb{H}_u) \\ &= E(e^{-\int_0^t r(\theta_s) ds} C(t, T, K, y, j, x) \mid \bar{\mathbb{L}}_u \vee \mathbb{H}_u) \end{aligned}$$

$$\begin{aligned}
&= E[e^{-\int_0^t r(\theta_s)ds} E(e^{-\int_t^T r(\theta_s)ds} S(x, K) \mid \bar{\mathbb{L}}_t \vee \mathbb{H}_t) \mid \bar{\mathbb{L}}_u \vee \mathbb{H}_u] \\
&= E[E(e^{-\int_0^T r(\theta_s)ds} S(x, K) \mid \bar{\mathbb{L}}_t \vee \mathbb{H}_t) \mid \bar{\mathbb{L}}_u \vee \mathbb{H}_u] \\
&= E[e^{-\int_0^T r(\theta_s)ds} S(x, K) \mid \bar{\mathbb{L}}_u \vee \mathbb{H}_u] \\
&= e^{-\int_0^u r(\theta_s)ds} E[e^{-\int_u^T r(\theta_s)ds} S(x, K) \mid \bar{\mathbb{L}}_u \vee \mathbb{H}_u] \\
&= V(t, T, K, y, j, x).
\end{aligned}$$

From Ito differential formula in Lemma 4.1, we have

$$dV(t, T, K, y_t, \theta_t, x(t)) = \mathcal{A}V(t, T, K, y_{t-}, \theta_{t-}, x(t^-))dt + \underbrace{\text{Martingale Terms}}.$$

As V is a martingale, the first term vanishes and (4.14), the following PIDE is obtained:

$$\begin{aligned}
&\frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(\theta_{s-})x_{s-} \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(\theta_{s-})x_s^2 \frac{\partial^2 V}{\partial x^2} \\
&+ \int_{|z| \leq 1} \left[V(s, T, K, y_s, \theta_s, x_{s-} + x_{s-}G(z, \theta_s)) - V(s, T, K, y_s, \theta_s, x_{s-}) \right. \\
&\quad \left. - G(z, \theta_s)x_{s-} \frac{\partial V}{\partial x} \right] \nu(\theta_s, dz) \\
&+ \int_{|z| > 1} [V(s, T, K, y_s, \theta_s, x_{s-} + x_{s-}H(z, \theta_s)) - V(s, T, K, y_s, \theta_s, x_{s-})] \nu(\theta_s, dz) \\
&+ \int_{z \in \mathbb{R}} \sum_{j \neq i} \lambda_{i,j}(y_s) V(s, T, K, y_s, j, x_{s-} e^z) \bar{b}(z \mid i, j) dz \\
&- V(s, T, K, y_{s-}, i, x_{s-}) \lambda_{\theta_{s-}, j}(y_s) = 0, \quad \forall t \in [0, T].
\end{aligned}$$

Hence, the proof is complete. \square

Definition 7.2. Let \tilde{C} be a continuous function on $[0, T] \times \mathbb{R}^+ \times \mathbb{R} \times [0, T] \times E \times \mathbb{R}^+$ into \mathbb{R}^+ representing the modified European call option price; let Υ be the characteristic function of \tilde{C} with respect to its third variable, and k denotes the logarithm of the positive real number K in Definition 7.1.

Remark 7.1. Assuming a deterministic interest rate r , a closed form formula for the Fourier transform of a modified vanilla European call option price is known Carr & Madan (1999). Let us denote C , \tilde{C} and η the European call price, the modified European call price of Carr and Madan type and a positive real number, respectively, for the payoff function of a European call option $S(x_T, K) = (x_T - K)^+ = (e^{\ln(x_T)} - e^k)^+$, with $k = \ln(K)$. Further assume that

$$\int_0^\infty |\tilde{C}(t, T, k, j, y)| dk < \infty, \quad \forall j \in E.$$

Then the modified European call price defined by Carr and Madan is expressed as follows:

$$\tilde{C}(t, T, k, y_t, \theta_t, x(t)) = e^\eta E_Q(e^{-\int_t^T r(s)ds} (e^{\ln(x_T)} - e^k)^+ | y_t, \theta_t, x_t). \quad (7.3)$$

In the following lemma, we recall the characteristic function of the modified European Carr and Madan type call option (Carr & Madan 1999).

Lemma 7.3. *If Υ and $\Psi(u, t, y_0, \theta_0, x(0))$ are the Fourier transform of \tilde{C} defined in Definition 7.2 and the conditional characteristic function of the log price process defined in Lemma 5.1, respectively, then we have*

$$\begin{aligned} & \Upsilon(T, w, y_0, \theta_0 = j, x_0) \\ &= \frac{e^{-\int_0^T r_s - ds}}{(\eta + iw)(1 + \eta + iw)} \Psi(wk - ix_0(1 + \eta), T, y_0, \theta_0 = j, x_0). \end{aligned} \quad (7.4)$$

Proof. The proof can be found in Carr & Madan (1999). □

Remark 7.2. We note that in the case of a regime switching interest rate Momeya (2012) uses the Carr and Madan type transformation to obtain the characteristic function of European call option prices. The formula in Carr & Madan (1999) is based on the characteristic function of occupation times which is known in closed form when market states are described by a Markov Chain. We have derived the characteristic function of the occupation times in Corollary 5.1 which allows us to extend the results in Elliott & Osakwe (2006) and Momeya (2012) when market states are described by a semi-Markov process.

In the context of a price process driven by the Brownian motion (Ghosh & Goswami 2009), an integral option price formula is obtained. In the following result, we present a similar pricing formula (Ghosh & Goswami 2009) in the context of (4.1), where we assume that f_s^j is the density of the increment of the log price process in an interval of length s , whenever the semi-Markov process is in state j for any $j \in I(1, m) = E$.

Lemma 7.4. *An integral option pricing formula in the context of model (4.1) is represented by the following formula:*

$$\begin{aligned} C(t, y_t, \theta_t, x_t) &= P(t, y_t, \theta_t) C^{\theta_t}(t, T, K, x_t) \\ &+ Q(t, y_t, \theta_t) \int_0^{T-t} e^{r(\theta_t)u} p(t, y_t, \theta_t) \left[\int_0^\infty \tilde{C}(t+u, 0, j, x_t) du \right] dx, \end{aligned} \quad (7.5)$$

with $P(t, y_t, \theta_t) = \frac{1-F(y_t+T-t|\theta_t)}{1-F(y_t|\theta_t)}$, $1 - P(t, y_t, \theta_t) = Q(t, y_t, \theta_t)$, $p(t, y_t, \theta_t) = \frac{f(y_t+T-t|\theta_t)}{1-F(y_t|\theta_t)}$ and $\tilde{C}(t+u, y_t, j, x_t) = \sum_{\theta_{t+u}=j, j \neq \theta_t} C(t+u, y_t, j, x_t) f_u^j(\ln(x/S_t))$, where $F(\cdot | \theta_{t-})$ and $f(\cdot | \theta_{t-})$ are defined in Remark 2.1. C^{θ_t} is the Black-Scholes

option price when the market is in state θ_t and $C(t, y_t, \theta_t, x_t)$ is short hand notation for $C(t, T, K, y_t, \theta_t, x_t)$.

Proof. The lemma follows by imitating the proof of Theorem 3.1 of Ghosh & Goswami (2009). Let $V(t) = v(t, y_t, \theta_t, x_t)$ defined as in Lemma 4.1, using the risk neutral pricing formula, the tower Law of expectations, the identity $1 = 1_{T_{n(t)+1} \leq 1} + 1_{T_{n(t)+1} > 1}$ and the notations

$$E[V(t) \mid s, y_s, \theta_s, x_s] = E_s[V(t)],$$

$$E[V(t) \mid s, y_s, \theta_s, x_s, T_{n(t)+1} < T] = E_s^{\leq T}[V(t)], t$$

$$E[V(t) \mid s, y_s, \theta_s, x_s, T_{n(t)+1} > T] = E_s^{> T}[V(t)]$$

$$E[V(t) \mid s, u, y_s, y_u, \theta_s, \theta_u, x_s, x_u] = E_{s,u}[V(t)]$$

$$E[V(t) \mid s, u, y_s, y_u, \theta_s, \theta_u, x_s, x_u, T_{n(t)+1} > T] = E_{s,u}^{> T}[V(t)]$$

$$E[V(t) \mid s, s + u, y_s, y_{s+u}, \theta_s, \theta_{s+u}, x_s, x_{s+u}, \tau_{n(s)} = y_t + u] = E_{s,u}^{\tau=T}[V(t)],$$

$\forall s, t, u \in \mathbb{R}_+$ and $s < t$, we obtain

$$\begin{aligned} & C(t, y_t, \theta_t, x_t) \\ &= E_{t-}[e^{\int_t^T r(\theta_s)ds} (x_T - K)^+] \\ &= E_{t-}[E[e^{\int_t^T r(\theta_s)ds} (x_T - K)^+ \mid y_t, \theta_t, x_t, T_{n(t)+1}]] \\ &= E_{t-}[1_{(T_{n(t)+1}) > T} E[e^{\int_t^T r(\theta_{s-})ds} (x_T - K)^+ \mid y_t, \theta_t, x_t, T_{n(t)+1}] \\ &\quad + E[1_{T_{n(t)+1} \leq T} E[e^{\int_t^T r(\theta_{s-})ds} (x_T - K)^+ \mid y_t, \theta_t, x_t, T_{n(t)+1}]] \\ &= P(T_{n(t)+1} > T \mid y_{t-}, \theta_{t-}, x_{t-}) E_{t-}^{> T}[E[C(t, y_{t-}, \theta_{t-}, x_{t-}) \mid T_{n(t)+1}]] \\ &\quad + P((T_{n(t)+1} \leq T) \mid y_{t-}, \theta_{t-}, x_{t-}) E_{t-}^{\leq T}[E[C(t, y_{t-}, \theta_{t-}, x_{t-}) \mid T_{n(t)+1}]] \\ &= P(T_{n(t)+1} > T \mid y_{t-}, \theta_{t-}, x_{t-}) E[C(t, y_{t-}, \theta_{t-}, x_{t-}) \mid T_{n(t)+1} > T] \\ &\quad + P(T_{n(t)+1} \leq T \mid y_{t-}, \theta_{t-}, x_{t-}) E[C(t, y_{t-}, \theta_{t-}, x_{t-}) \mid T_{n(t)+1} \leq T] \\ &= P(T_{n(t)+1} > T \mid y_{t-}, \theta_{t-}, x_{t-}) C^{\theta_t}(t, x_t) + P(T_{n(t)+1} > T \mid y_{t-}, \theta_{t-}, x_{t-}) \\ &\quad \times E_{t-}^{\leq T}[E(C(t, y_{t-}, \theta_{t-}, x_{t-}) \mid \tau_{n(t)} = y_t + u)] \\ &= P(t, y_t, \theta_t) C^{\theta_t}(t, x_t) + Q(t, y_t, \theta + t) \\ &\quad \times \int_0^{T-t} p(T - u, y_t, \theta_t) E[C(t, y_{t-}, \theta_{t-}, x_{t-}) \mid \tau_{n(t)} = y_t + u] du \\ &= P(t, y_t, \theta_t) C^{\theta_t}(t, x_t) + Q(t, y_t, \theta_t) \int_0^{T-t} p(t, y_t, \theta_t) E_{t+u,t} \\ &\quad \times [e^{r(\theta_t)u} E[e^{\int_{t+u}^T r(\theta_{s-})ds} (x_T - K)^+ \mid y_{t-}, \theta_{t-}, x_{t-}, \tau_{n(t)} = y_t + u]] \end{aligned}$$

$$\begin{aligned}
 &= P(t, y_t, \theta_t) C^{\theta_t}(t, x_t) + Q(t, y_t, \theta_t) \int_0^{T-t} E_t[e^{r(\theta_t)u} p(T-u, y_t, \theta_t) \\
 &\quad \times E_{t,u}^{\tau=y_t+u}[e^{\int_{t+u}^T r(\theta_s^-) ds} (x_T - K)^+]] du \\
 &= P(t, y_t, \theta_t) C^{\theta_t}(t, x_t) + Q(t, y_t, \theta_t) \int_0^{T-t} e^{r(\theta_t)u} p(T-u, y_t, \theta_t) du \\
 &\quad \times \int_0^\infty \sum_{\substack{\theta_{t+u}=j \\ j \neq \theta_t}} E[C(t+u, y_{t+u}=0, \theta_{t+u}, x_t^-) | \tau_{n(t)} = y_t + u] \\
 &\quad \times f_u^j(\ln(x/S_t)) du dx \\
 &= P(t, y_t, \theta_t) C^{\theta_t}(t, x_t) + Q(t, y_t, \theta_t) \int_0^{T-t} e^{r(\theta_t)u} p(T-u, y_t, \theta_t) \\
 &\quad \times \int_0^\infty \tilde{C}(t+u, 0, j, x_t) du dx. \quad \square
 \end{aligned}$$

In the following section, we provide a numerical application to exhibit the role, scope and significance of the developed results. Furthermore, based on the options on DJIA quoted on March 3, 2008, we also present calibration results. The data set was retrieved from Deville (2007).

8. Numerical Applications

We present simulated European style call option prices and implied volatilities derived from option prices generated by a semi-Markov switching exponential Lévy process.

8.1. Simulations

Option prices derived in our framework are more flexible than option prices derived from the more commonly used Markov regime switching exponential Lévy processes. We also demonstrate that prices simulated in our framework produce implied volatilities consistent with stylized facts of the option market, namely, the smile and the smirk. The risk neutral measure considered in this section is the conditional minimum entropy martingale measure developed in Sec. 6. Simulations of option prices and implied volatilities will be performed from semi-Markov regime switching Black-Scholes (SMBS), semi-Markov regime switching Merton jump diffusion (SMMJD) and semi-Markov regime switching normal inverse Gaussian (SMNIG) processes. The choice of these three Lévy processes is rooted in the fact that they highlight three important classes of Lévy processes: Diffusion processes, jump diffusion processes and processes with infinite activity. At each market state we assume j , $H(j, z) = G(j, z) = z$ in (4.2) and the SMBS has no Lévy jump component, hence its Lévy measure is $\nu(j, \cdot) = 0, \forall j \in E$. The diffusion coefficient at each state j is

$\sigma(j), \forall j \in E$. At each state j , the SMMJD has Lévy jumps arriving at a Poisson rate with mean $\eta(j)$; its jumps size is normally distributed with mean and standard deviation $\rho(j)$ and $\delta(j)$, respectively for any state $j \in E$. From Tankov (2003), it's Lévy measure is

$$\nu(j, dz) = \frac{\eta(j)}{\delta(j)\sqrt{2\pi}} \exp\left[-\frac{1}{2} \frac{(z - \rho(j))^2}{\delta^2(j)}\right] dz.$$

At each state $j \in E$, the SMNIG is a time changed Brownian motion with drift rate $\mu(j)$. The new time scale is defined by an inverse Gaussian process with parameters $\delta(j)$ and $\rho^2(j) - \eta^2(j)$. The SMNIG is a pure jump process and consequently does not have diffusion component ie $\sigma(j) = 0, \forall j \in E$. When the market is in state $j \in E$, the characteristic function of the SMNIG, is expressed as follows:

$$\phi(j, u) = \exp[-\delta\sqrt{\rho^2(j) - (\eta(j) + iu)^2} - \sqrt{\rho^2(j) - \eta^2(j)}], \quad \text{with } i = \sqrt{-1}.$$

Under the risk neutral measure, the price jumps at times of regime changes $\kappa_{i,j}$ are assumed to follow the distributions:

$$\kappa_{i,j} = \begin{cases} 1 - \varepsilon_{i,j} & \text{with probability } p_{i,j} \\ 1 & \text{with probability } 1 - p_{i,j} - q_{i,j} \\ 1 + \varepsilon_{i,j} & \text{with probability } q_{i,j}. \end{cases} \quad (8.1)$$

The risk neutral condition (6.28) requires that $E_{P^{\alpha^*}}(\beta_n | \theta_{n-1}, \theta_n) = 1, \forall n \in I(1, \infty)$. This implies that

$$p_{i,j} = q_{i,j}, \quad \forall (i, j) \in E^2. \quad (8.2)$$

From (8.1) and (8.2), the characteristic function of $\ln(\kappa_{i,j})$ reads

$$E[e^{iu \ln(\kappa_{i,j})}] = [p_{i,j}(e^{-iu \ln(1+\varepsilon_{i,j})} + e^{iu(1-\ln \varepsilon_{i,j})} - 2) + 1], \quad (8.3)$$

$$\varepsilon_{i,j} \in [0, 1], \quad p_{i,j} \in \left[0, \frac{1}{2}\right], \quad \forall (i, j) \in E^2,$$

where the coefficient of u is $i = \sqrt{-1}$ and the subscript " i " of κ , ε and p stand for the i th state of the semi-Markov process. The sojourn time distribution of the semi-Markov process is assumed to be a piecewise exponential distribution approximating a two-parameter Weibull distribution. It is worth noting that piecewise exponential distributions are known to have piecewise constant intensities. Knowing that Riemann integrable functions can be approximated almost everywhere by step functions, we can simulate prices in our framework with a wide range of sojourn times. Let T^* be the time horizon of the market, with $T < T^*$, where T is any option contract maturity time. We consider the following partition $0 = a_0 < a_1 < \dots < a_{M-1} = T^*$ of $[0, T^*]$ and we denote $a_M = \infty$. From the definition of the Weibull distribution and (2.9), the Weibull intensity function can be

approximated by the following piecewise constant semi-Markov intensity function associated with the partition $(a)_{k=0}^{M-1}$

$$\begin{aligned} \lambda_{i,j}(y_s) &= \begin{cases} p_{i,j} \sum_{k=0}^{M-1} \frac{\vartheta_i}{\varsigma_i} \left(\frac{a_k^*}{\varsigma_i} \right)^{\vartheta_i-1} 1_{[a_k, a_{k+1})}(y_s) & \text{if } i \neq j \\ - \sum_{j=1, j \neq i}^m \lambda_{i,j}(y_s) & \text{otherwise} \end{cases} \\ &= \begin{cases} \alpha_{i,j} \sum_{k=0}^{M-1} (a_k^*)^{\vartheta_i-1} 1_{[a_k, a_{k+1})}(y_s) & \text{if } i \neq j \\ - \sum_{j=1, j \neq i}^m \lambda_{i,j}(y_s) & \text{otherwise,} \end{cases} \end{aligned} \quad (8.4)$$

$\forall s \in [0, T]$, with $\alpha_{i,j} = p_{i,j} \frac{\vartheta_i}{\varsigma_i}$ and $a_k^* = \frac{a_k + a_{k+1}}{2}$, $\forall k \in I(0, M-2)$. Figure 2 shows that for the particular parameters chosen, the piecewise constant approximation of the Weibull intensity is quite close to the actual Weibull intensity. In a market with Markov regimes, intensities are assumed to be constant. This is quite a restrictive assumption, since it implies that as time goes by, the market propensity for changing regimes stays constant even when market conditions change. It is indeed more realistic to allow the intensity matrix to be time dependent and vary across the lifespan of the derivative contract. This is precisely the advantage of semi-Markov over Markov market regimes, as semi-Markov intensities depend on the time spent on the current state. It is for instance well known that markets pre and post 2007 financial crisis are quite different as the volatility has gone up a notch and prices have become more erratic Manda (2010). We therefore expect switching rates between market regimes in the post crisis market to be different. Option price simulations are performed through inversion of the Fourier transform of the modified option price in Carr & Madan (1999). Inversion is performed via the fractional Fourier transform described in Chourdakis (2004), as it allows a greater flexibility in the choice of the log strike grid parameter and the frequency grid parameter. The size of the log strike grid and the frequency grid is set at $N = 4096$. The infinite integral in the frequency domain is truncated at 512 and the domain of $\ln(\frac{K}{x_T})$ is set to be $[-1, +1]$. Under this setting, the fractional parameter is $\epsilon = \frac{1}{2\pi} \times \frac{512}{4096} \times \frac{2}{4096} \approx 9.72 \times 10^{-6}$. Throughout this section, we assume that the market has $m = 4$ distinct regimes, $x_T = \$100$, the interest rate is constant across all states $r = (0.05, 0.05, 0.05, 0.05)$, the initial backward recurrence time $y = 0.1$ year, the time horizon of the market is $T^* = 5$ years, the time to maturity is set to $T = 0.5$ year for option price simulations and $T \in [0, 1.5]$ for “Implied volatility” surface simulations. The parameters specific to our framework, not accounted for in markets with Markov regimes are the backward recurrence time y at time of pricing and the shape parameter vector $(\vartheta_i)_{i \in E}$ of the sojourn time distributions. It is therefore of interest to see

Table 1. Parameter values used for simulation of Markov versus semi-Markov option prices.

α	ϑ	ε	p
$\begin{pmatrix} -0.4 & 0.3 & 0.1 & 0 \\ 1 & -0.4 & 0.3 & 0 \\ 0.2 & 0.4 & -2.5 & 1 \\ 0.5 & 0.5 & 1 & -2 \end{pmatrix}$	$(0.3 \ 0.1 \ 0.2 \ 0.05), (1 \ 1 \ 1), (2 \ 3 \ 8 \ 5)$	$\begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.05 & 0.05 & 0.3 \\ 0.4 & 0.3 & 0.2 \end{pmatrix}$	$\begin{pmatrix} 0.2 & 0.3 & 0.25 \\ 0.2 & 0.35 & 0.2 \\ 0.35 & 0.3 & 0.1 \\ 0.1 & 0.15 & 0.25 \end{pmatrix}$
σ $(0.3 \ 1 \ 0.5 \ 0.11)$	η $(0.1 \ 0.3 \ 0.9 \ 0.2)$	δ $(-0.9 \ 0.3 \ -0.5 \ -0.1)$	ρ $(0.2 \ 0.1 \ 0.2 \ 0.1)$

Table 2. Parameter values used for simulation of the SMBS volatility surface.

α	ϑ	ε	p
$\begin{pmatrix} -0.9 & 0.3 & 0.1 & 0.5 \\ 0.1 & -0.6 & 0.3 & 0.2 \\ 0.2 & 0.4 & -0.7 & 0.1 \\ 0.4 & 0.2 & 0.4 & -1 \end{pmatrix}$	$(5 \ 0.2 \ 4 \ 0.5)$	$\begin{pmatrix} 0.1 & 0.05 & 0.2 \\ 0.3 & 0.2 & 0.1 \\ 0.25 & 0.35 & 0.1 \\ 0.05 & 0.6 & 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.15 & 0.1 & 0.1 \\ 0.05 & 0.3 & 0.1 \\ 0.23 & 0.2 & 0.2 \\ 0.05 & 0.25 & 0.2 \end{pmatrix}$
σ $(0.3 \ 0.5 \ 0.6 \ 0.7)$	η $(0.02 \ -0.03 \ -0.5 \ -0.1)$	δ $(0.1 \ 0.2 \ 0.1 \ 0.09)$	ρ

Table 3. Parameter values used for simulation of the SMMJD volatility surface.

α				ϑ				ε				p			
$\begin{pmatrix} -0.65 & 0.1 & 0.05 & 0.5 \\ 1 & -1.35 & 0.25 & 0.1 \\ 1 & 0.6 & -1.8 & 0.2 \\ 0.1 & 2 & 0.8 & -2.9 \end{pmatrix}$				$(5 \quad 0.2 \quad 4 \quad 0.5)$				$\begin{pmatrix} 0.1 & 0.05 & 0.2 \\ 0.3 & 0.2 & 0.1 \\ 0.25 & 0.35 & 0.1 \\ 0.05 & 0.6 & 0.1 \end{pmatrix}$				$\begin{pmatrix} 0.25 & 0.4 & 0.1 \\ 0.25 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.2 \\ 0.05 & 0.4 & 0.2 \end{pmatrix}$			
σ				η				δ				ρ			
$(0.2 \quad 0.3 \quad 0.35 \quad 0.3)$				$(1 \quad 3 \quad 0.96 \quad 1)$				$(0.02 \quad -0.03 \quad -0.5 \quad -0.1)$				$(0.1 \quad 0.2 \quad 0.1 \quad 0.09)$			

Table 4. Parameter values used for simulation of the SMNIG volatility surface.

α	ϑ				ε				p			
$\begin{pmatrix} -1 & 0.3 & 0.1 & 0.6 \\ 0.2 & -1.9 & 0.7 & 1 \\ 0.25 & 0.1 & -0.95 & 0.6 \\ 0.5 & 0.2 & 0.1 & -0.8 \end{pmatrix}$	$(5 \quad 0.2 \quad 4 \quad 0.5)$				$\begin{pmatrix} 0.1 & 0.05 & 0.2 \\ 0.3 & 0.2 & 0.1 \\ 0.25 & 0.35 & 0.1 \\ 0.05 & 0.6 & 0.1 \end{pmatrix}$				$\begin{pmatrix} 0.25 & 0.4 & 0.1 \\ 0.25 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.2 \\ 0.05 & 0.4 & 0.2 \end{pmatrix}$			
σ	η				δ				ρ			
	$(0.15 \quad 0.1 \quad 0.12 \quad 0.2)$				$(1.2 \quad 1.8 \quad 1.8 \quad 2)$				$(-1 \quad -1.5 \quad 1.4 \quad 1)$			

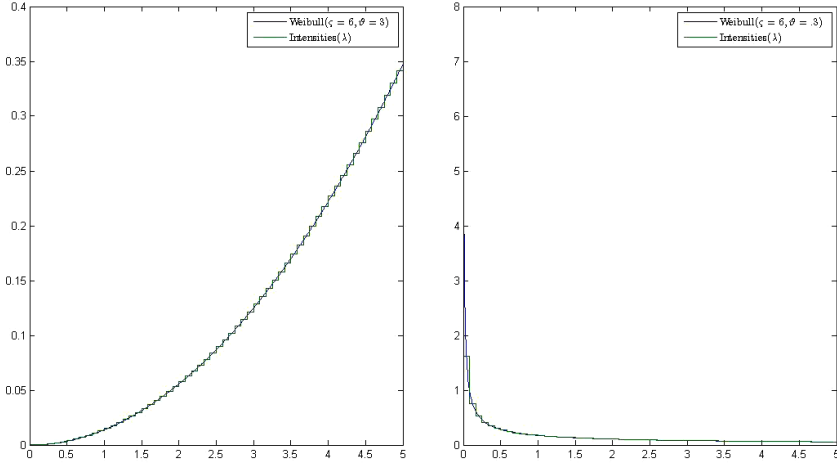


Fig. 1. Weibull intensity versus sojourn time distribution intensity.

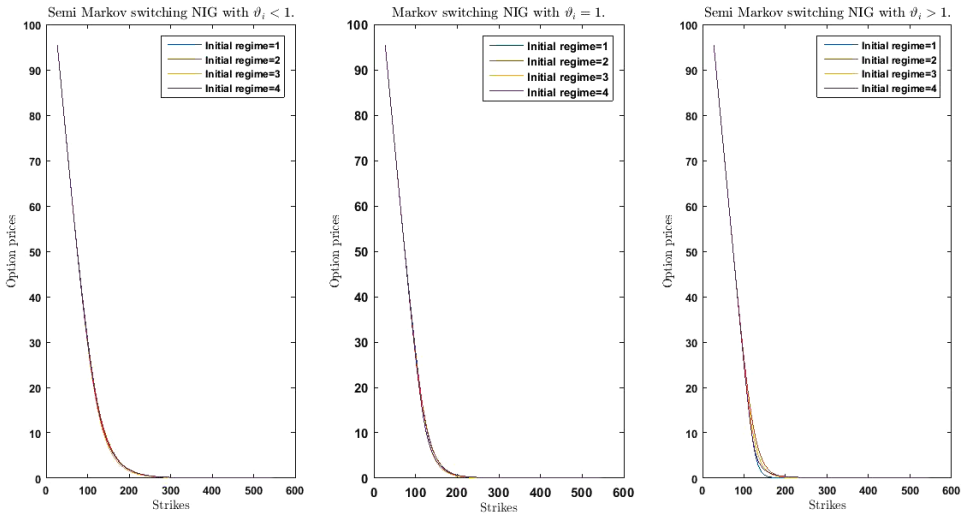


Fig. 2. Semi-Markov regime switching NIG option prices at all market regimes for different shape parameters values.

how much difference those parameters induce when option prices from a market with semi-Markov regimes are compared with option prices from a market with Markov regimes *ceteris paribus*. The partition associated with the piecewise constant semi-Markov conditional intensity considered (8.4) is a regular grid of $[0, T^*]$ where $M = 61$ with $a_k = k \frac{T^*}{M-1}, k \in I(0, M-1)$. Figure 1 shows how well λ approximates weibull intensities. This choice of M is rooted in the fact that financial performances of firms are usually made available on a quarterly basis. We therefore consider three

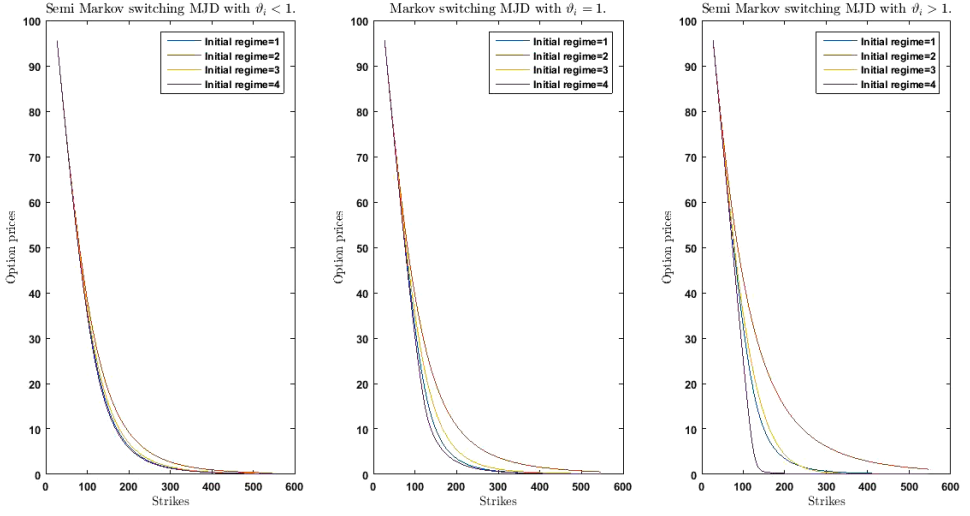


Fig. 3. Semi-Markov MJD option prices at all market regimes for different shape parameters values.

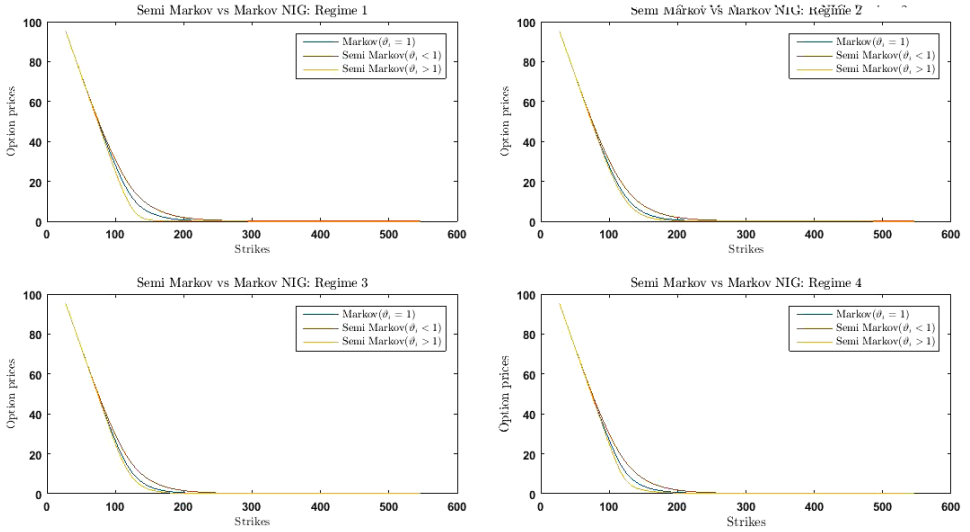


Fig. 4. Semi-Markov NIG option prices at all market regimes for different shape parameters values.

periods within each quarter: The earning month where the market likely (over)reacts to reports, the post earning month with a possible market correction and the pre earning month where the market is preparing for the next earning report. The intensity of the semi-Markov process θ_t of Definition 2.1 is assumed constant within each month. Parameter values chosen for simulations of option prices and volatility surfaces are summarized in Tables 1–4, respectively. Option price simulations from

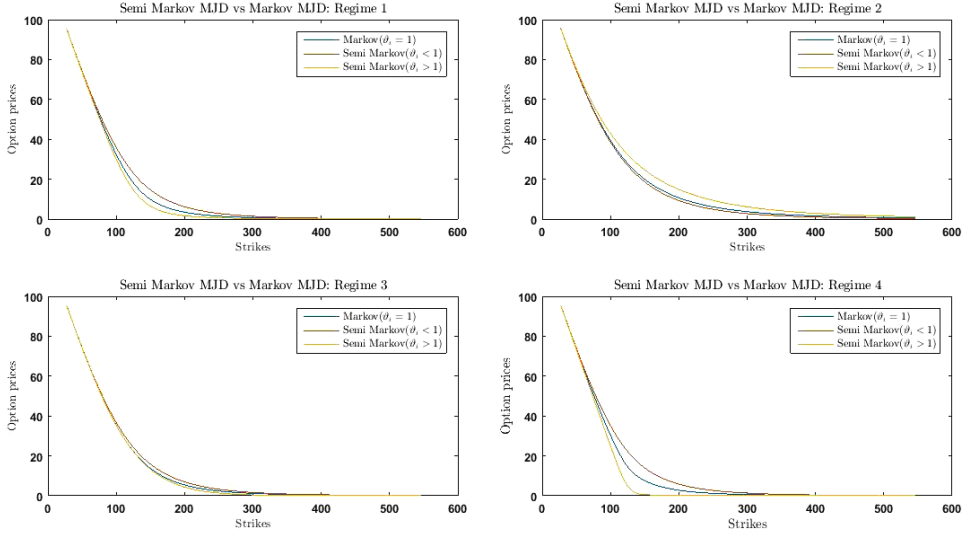


Fig. 5. Comparison of simulated European call option prices derived from Markov and semi-Markov switching MJD models.

Markov regime switching models and semi-Markov switching model with increasing intensities are performed using $\vartheta_i = (1, 1, 1, 1)$ and $\vartheta_i = (2, 3, 8, 5)$, respectively. In Figs. 2 and 3, one notices that deep in-the-money option prices for $\vartheta_i < 1, \forall i \in I(1, m)$ and $y = 0.1$ year flirt with the \$120 mark, while prices induced by $\vartheta_i > 1$ and $\vartheta_i = 1, \forall i \in I(1, m)$ lie below \$100. In the presented simulated prices, it appears that the initial backward recurrence time and the shape parameter of sojourn time distribution do induce noticeable price differences between options in Markov and semi-Markov market states. We also note price differences between market states as shown in Figs. 2 and 3. On the other hand, we note that by changing the shape parameter of the sojourn time distribution, semi-Markov prices prove to be more exible than Markov prices as shown in Figs. 4 and 5. One recalls that the higher the shape parameter values, the smaller the mean sojourn time of the market at each regime and the smaller the regime change risk. This partially explains why deep in-the-money options in Figs. 4 and 5 are cheaper as ϑ gets larger. As far as implied volatilities are concerned, smiles, smirks and other shapes are seamlessly reproduced as evidenced in Fig. 6.

8.2. Calibration

The Dow Jones Industrial Average Index (DJIA) is a performance indicator of 30 large public US companies that accounts for approximately 20% of the U.S. market. We consider options quotes on the Dow Jones Industrial Average Index (DJX) on March 3, 2008, as the global financial crisis was in full effect. We calibrate the parameters of our models (SMBS and Markov BS) to market option prices in

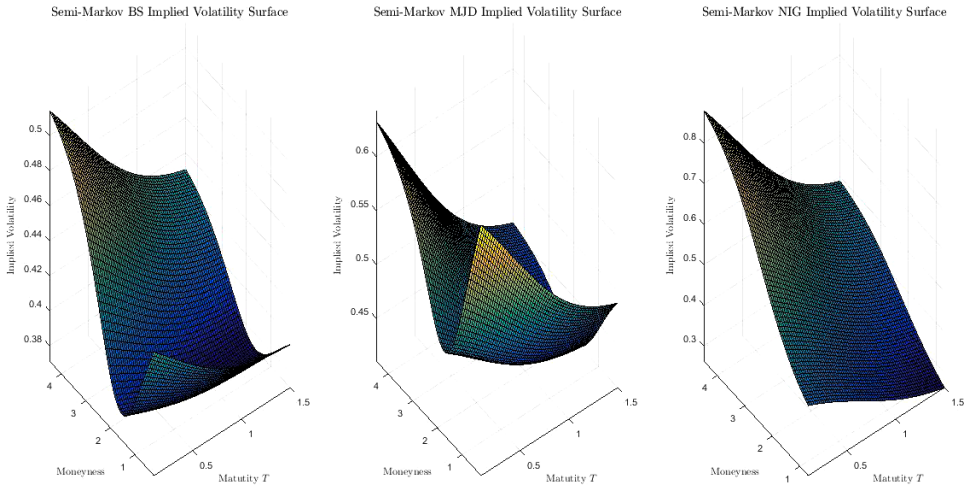


Fig. 6. Implied volatility surfaces from option prices generated by semi-Markov switching BS, MJD and NIG respectively at the first market state.

Table 5. We use residual sum of squares (RSS) to compare the fit of both models. The model parameters sought are chosen so as to minimize the sum of squared differences between market options and model option prices. We assume that $m = 2$; the option contract has maturity time $T = 47$ days; the underlying stock does not pay dividends and is currently valued at $S = \$122$; the sojourn time intensities are assumed to be a piecewise constant approximation of Weibull intensities as defined in (8.4); the fractional parameter is given in Sec. 8.1 and the current backward recurrence time is assumed to be $y = 252$ days. Calibration results obtained are

Table 5. DJX option quotes of March 3, 2008.

Strike price	Strike S/K	Prices ($T = 47$)
98	1.25	24.43
99	1.23	23.40
100	1.22	22.50
101	1.21	21.55
102	1.20	20.63
103	1.19	19.68
104	1.18	18.75
105	1.16	17.83
106	1.15	16.90
107	1.14	15.98
108	1.13	15.10
109	1.12	14.23
110	1.11	13.33
111	1.10	12.45
112	1.09	11.63
113	1.08	10.78

Table 5. (*Continued*)

Strike price	Strike S/K	Prices ($T = 47$)
114	1.07	9.95
115	1.06	9.18
116	1.05	8.40
117	1.04	7.68
118	1.04	6.93
119	1.03	6.23
120	1.02	5.58
121	1.01	4.95
122	1.00	4.35
123	0.99	3.80
124	0.99	3.25
125	0.98	2.74
126	0.97	2.28
127	0.96	1.90
128	0.95	1.52

Table 6. Markov BS and SMBS parameter estimates.

Model	$(\hat{\alpha}_0, \hat{\alpha}_1)$	$(\hat{\vartheta}_0, \hat{\vartheta}_1)$	$(\hat{\varepsilon}_0, \hat{\varepsilon}_1)$	(\hat{p}_0, \hat{p}_1)	$(\hat{\sigma}_0, \hat{\sigma}_1)$	RSS
SMBS	(70.85, 1.523)	(26, 0.202)	(0.008, 0.55)	(0.001, 0.02)	(0.212, 0.18)	2.07
Markov BS	(79.03, 9.363)	(1, 1)			(0.176, 0.175)	3.73

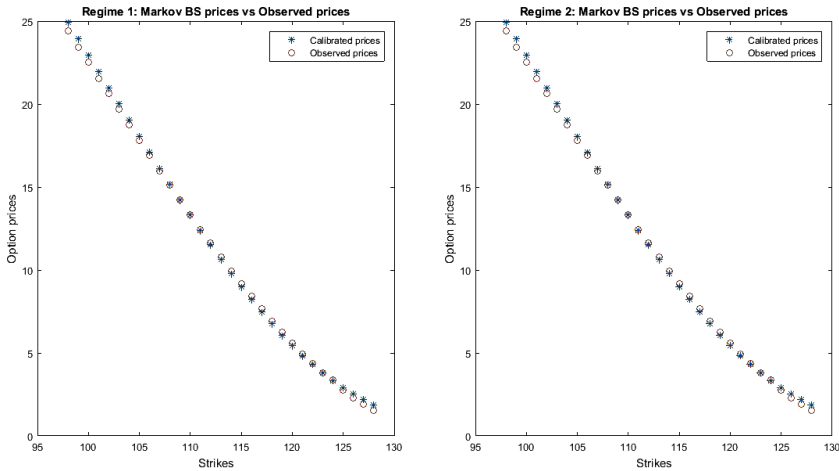


Fig. 7. Markov BS: Calibrated versus observed prices.

summarized in Table 6 and plotted in Figs. 7 and 8. One note that the RSS is considerably lower for semi-Markov regime switching models. In fact, the RSS is reduced by 44%. The fit of SMBS is visibly better as shown in Figs. 7 and 8. Moreover, the scale parameters of the Weibull shows that market option prices are best explained by semi-Markov market regimes with $\vartheta_1 = 26$, $\vartheta_2 = 0.202$. These

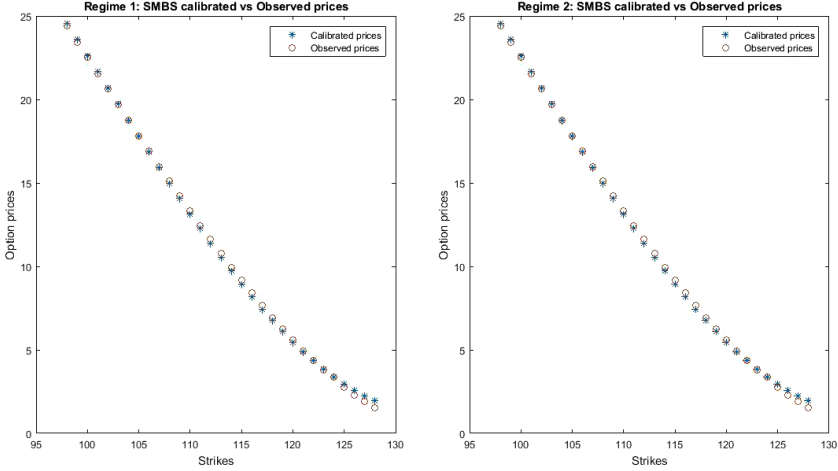


Fig. 8. SMBS: Calibrated versus observed prices.

shape parameter values show that state 1 and 2 have increasing and decreasing intensities. More precisely, the longer the market stays in state 1, the more likely it is to switch to state 2 and the longer the market stays in state 2 the more likely it is to stay in state 2, respectively. A look at the discrete jump size also shows quite a significant risk neutral probability (2%) of drop or jump in price of more than 50% when the market is in state 2. This is in line with the widespread fear and the observed erratic market prices during the financial crisis. Indeed, on September 29, 2008, the DJIA suffered its biggest one-day price drop (7%) and a few month later, November 13, 2008, the market registered its biggest one-day price jump (11%). More importantly, the Dow Jones suffered a whopping 33% drop in 2008.

9. Conclusion

In this work, we proposed a theoretical framework for asset price models by developing a closed form an exponential Lévy under a semi-Markov switching process. In addition, we established a few distributional properties of the price process in (4.1). We derived an infinitesimal generator of the triplet (y_t, θ_t, x_t) and the characteristic function of the log price process L_t^θ which allowed us to get a closed form expression for the characteristic function of plain vanilla European style call option prices. Such a characteristic function could prove to be useful in the calibration of model parameters to market options prices through inverse Fourier transform, as demonstrated in Sec. 8. Two problems of interest are currently being investigated: (1) an evaluation of the effects of the backward recurrence time, the sojourn time distribution and the jumps distribution on option prices induced by semi-Markov switching asset price models; and (2) the pricing of exotic options in semi-Markov market regimes.

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